Purely Axial Torsion Waves – New Solutions of Metric-affine Gravity

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4th June 2015

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Structure of talk

- Mathematical model
- PP-waves with axial torsion
- New vacuum solutions of quadratic metric-affine gravity
- Physical interpretation
Metric-affine gravity

Spacetime is a connected real 4–manifold $M$ with a Lorentzian metric $g$ and an affine connection $\Gamma$. The field equations
\[ \nabla_\mu u_\lambda = \partial_\mu u_\lambda + \Gamma^\lambda_\mu\nu u_\nu. \]

An independent linear connection $\Gamma$ distinguishes MAG from GR - $g$ and $\Gamma$ viewed as two totally independent quantities.

Action is $S = \int q(R)$, where $q(R)$ is a Lorentz invariant purely quadratic form on curvature.

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Field equations

Euler–Lagrange equations

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\frac{\partial S}{\partial g} = 0, \quad (1)
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Euler–Lagrange equations

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The Yang–Mills action for the affine connection is a special case

\[ q(R) := R^\kappa_{\lambda \mu \nu} R^\lambda_{\kappa \mu \nu}. \]
Classical pp-waves

Definition
A pp-wave is a Riemannian spacetime which admits a non-vanishing parallel spinor field ($\nabla \chi = 0$).

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$$\text{d}s^2 = 2\text{d}x_0 \text{d}x_3 - (\text{d}x_1)^2 - (\text{d}x_2)^2 + f(x_1, x_2, x_3) (\text{d}x_3)^2$$

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in some local coordinates.

Well known spacetimes in GR, simple formula for curvature.
Generalised pp-waves

A generalised pp-wave with purely axial torsion is a metric compatible spacetime with pp-metric and torsion

\[ T = A^a (\phi) \]

where \( A \) is a real vector field, \( \phi : M \rightarrow \mathbb{R} \), and \( \phi(x) = \int_M l \cdot dx \).

Torsion (3) clearly purely axial.
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A a real vector field \( A = k(\phi) \cdot l \), where \( k : \mathbb{R} \to \mathbb{R} \) arbitrary

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\[ R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) + \frac{1}{4}k^2 \text{Re}\left( (l \wedge m) \otimes (l \wedge m) \right) - \frac{1}{2}k' \text{Im}\left( (l \wedge m) \otimes (l \wedge m) \right) \]

- Torsion of a generalised pp-wave is

\[ T = -\frac{i}{2} kl \wedge l \wedge m \wedge m \]

- Ricci curvature is

\[ \text{Ric} = \frac{1}{2}(f_{11} + f_{22} - k^2) l \otimes l \]
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Generalised pp-waves with purely axial torsion of parallel $\{\text{Ric}\}$ are solutions of the system of equations (1), (2).
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- Note that by \(\{Ric\}\) we denote the Riemannian Ricci curvature.
- Condition \(\{\nabla\}\{Ric\} = 0\) implies that \(f_{11} + f_{22} = C\).
- Result also holds if \(Ric\) is assumed to be parallel.
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Generalised pp-waves with purely axial torsion of parallel \( \{Ric\} \) are solutions of the system of equations (1), (2).
Curvature of generalised pp-waves is split. Torsion and torsion generated curvature are waves traveling at the speed of light. Underlying pp-space can be viewed as the ‘gravitational imprint’ created by wave of some massless field. Mathematical model for some massless particle?
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Mathematical model for some massless particle?
Metric-affine vs Einstein-Weyl
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Look at massless Dirac (or Weyl) action

\[ S_W := 2i \int \left( \xi^a \sigma^\mu_{\ ab} (\nabla_\mu \xi^b) - (\nabla_\mu \xi^a) \sigma^\mu_{\ ab} \xi^b \right), \]
Metric-affine vs Einstein-Weyl

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In a generalised pp-space Weyl’s equation takes form

\[ \sigma^\mu_{ab} \{\nabla\}_{\mu} \xi^a = 0. \]
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\[ S_{EW} := k \int \mathcal{R} + S_{\text{neutrino}}, \]

\[ \partial S_{EW}/\partial g = 0, \]

\[ \partial S_{EW}/\partial \xi = 0. \]
Thank You very much and welcome to Tuzla!