Vacuum Solutions for Metric–affine Gravity with Spectral Analysis of the Massless Dirac Operator

submitted by

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Summary

In this thesis we deal with quadratic metric-affine gravity and the massless Dirac operator. We review known solutions for quadratic metric-affine gravity and we present new non-Riemannian solutions for this theory. The new solution is presented in the form of generalised pp-waves with purely axial torsion. We also propose a physical interpretation of these new solutions by comparing them to solutions of the Einstein-Weyl theory.

Another aim of this thesis is the spectral analysis of the massless Dirac operator on a closed 3-dimensional manifold. The massless Dirac operator describes a massless neutrino. We review two examples where the spectrum of this operator can be evaluated explicitly and it turns out that in these two particular examples the spectrum is symmetric about zero, although there is no mathematical or physical reason for it to be symmetric in the general case. Applying perturbation theory methods to the massless Dirac operator, we successfully observe spectral asymmetry on the 3-torus and derive explicit asymptotic formulae for perturbations of the eigenvalues ± 1.

Chapter 1

Introduction

The first developments in the theory of gravity go back to ancient Greece. In those times, the motion of the body in a free fall was considered purely philosophically and much time had passed before any mathematical model was set. Aristotle presented a philosophical argument that heavier bodies must fall faster than lighter ones and for centuries no one doubted this argument.

Doubts that this theory of Aristotle was true arose during the Renaissance and were presented by the great Italian scientist Galileo Galilei. Galileo first began an analysis and experimental verification of the laws of movement and presented these results in his two works *Dialogo* [37] in 1632 and *Discorsi* [38] in 1638. From those observations Galileo concluded that two bodies under the influence of the gravitational field of the Earth move in the same way and independently of their respective masses. This was in stark contrast to Aristotle's arguments.

The greatest contribution after Galileo in the research of the theory of gravity was given by the great English scientist Sir Isaac Newton. Newton based his observations on Kepler's laws of planetary motion and in 1687 he presented his observations in his famous work *Philosophiae Naturalis Principia Mathematica* [66]. The basic premise of his work was that the geometry of the three-dimensional space is Euclidean in nature and time is considered as a separate parameter which is required for the description of movement. Newton's theory of gravity claimed that the mass of the body is the source of gravity and that any two bodies attract each other with the force that is proportional to the product of their masses and inversely proportional to the square of their distance. This theory of gravity was in accordance with the Galilean principle of equivalence, but the principle itself had no effect on the construction of Newton's theory.

However, as one of the fundamental laws of physics, Newton's theory of gravity had defects in the form of time invariance and depended only on spatial distance. According to Newton's theory of gravity, the strength of the gravitational force between two objects is proportional to the inertial masses of the objects. The inertial mass has a dual role: it is a measure of the resistance of the object to the change of its speed, but on the other hand it also plays the role of the gravitational charge. In the same way as an electric charge determines the strength of the electric force between two charged objects, in Newton's theory the inertial mass determines the strength of the gravitational force. The second problem was that the gravitational force was described as the force of attraction between two objects. Consequently, if one was to move one of these two objects, the other object would 'realise' that fact due to the changes in the gravitational force regardless of their distance. The third problem was the discrepancy between the predictions of Newton's theory and experimental indicators of the precession of the orbit of Mercury.

1.1 General relativity and the alternatives

Newton's theory of gravity was further improved upon by Albert Einstein with his *theory of general relativity*, see [27]. The entire concept of Newton's understanding of space was changed by Einstein and space and time were no longer considered separately. The theory of general relativity is a theory of gravity that Einstein developed between 1907 and 1915 using Riemannian geometry. Einstein's colleague from his student days, the mathematician Marcel Grossmann, together with Hermann Minkowski, helped Einstein in the formulation of this theory. The new concept of *spacetime* of general relativity is fully described by the metric and this metric does not define only the distance but also the parallel transport, i.e. the Levi-Civita connection. As a central part of his theory of gravity, Einstein proposed the equation that relates geometry and matter

$$\underbrace{Ric_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}}_{\text{geometry}} = \underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{\text{matter}},$$

where $T_{\mu\nu}$ is the stress energy tensor which depends on matter, G is the gravitational constant, c is the speed of light, Ric is the Ricci curvature (1.18) and $\mathcal{R}$ is the scalar curvature (1.19). This equation presents a direct connection between the geometry of spacetime and matter. The vacuum Einstein equation has the form

$$Ric_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 0.$$

General relativity could be interpreted as follows: *spacetime tells matter how to move and matter tells the spacetime how to curve*, see [104]. Matter 'curves' spacetime, so in addition to the changes of spatial coordinates, something also happens to the time coordinate in the spacetime continuum. General relativity predicts that the time near some massive objects is more 'bent', i.e. time is slower near objects with greater mass than near objects with lesser mass. General relativity has been confirmed by much experimental data, such as the precession perihelion of Mercury, the deflection of light as it passes close to the Sun, the gravitational red shift, the time delay of the radar signal, the operation of the GPS device, etc. General relativity predicted the existence of gravitational waves, which was recently experimentally confirmed by the Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016. General relativity predicted the existence of black holes as objects in the spacetime continuum whose gravity is so strong that any form of matter can not escape them, even light which is moving as fast as it possible in nature. Modern cosmological observations support the natural existence of such objects in space. A comprehensive review of spacetimes in general relativity was done by Griffiths and Podolský [48].

Einstein himself expected much more from his theory of gravity. He was not fully satisfied because it failed to merge the gravitational field and the electromagnetic field into a single model, see [28]. In order to relate gravity and electromagnetism, Einstein thought about an alternative theory of gravity known as *teleparallelism*, see [97]. The teleparallel spacetime is seen as the connected real 4-dimensional manifold equipped with a Lorentzian metric whose curvature vanishes, but whose torsion does not vanish. A metric tensor is defined by the components of a dynamic tetrad field. For more results in the field of this alternative theory of gravity see e.g. [23, 34, 64, 69, 86, 95, 96, 102].

Gravity is the only physical interaction for which there is no consistent quantum formulation. Many attempts of the quantisation of gravity have thus far been unsuccessful. There are many alternative theories such as *metric-affine gravity, gauge symmetry, supergravity, Kaluza-Klein theory* or *string theory* that may lead us to a unified quantum theory of the fundamental interactions. Previous successes of these ideas motivate us to study them in more detail and hopefully it will bring us closer to the formulation of quantum gravity. General relativity has solved many disadvantages of earlier theories, but we still have many open questions which led to the development of alternative theories of gravity. In this thesis, we are dedicated to the study of an alternative theory of gravity known as *metric-affine gravity.*

Metric-affine gravity (MAG) is an alternative theory of gravity which is a extension of general relativity. In metric affine-gravity we leave the Riemannian spacetime of Einstein's general relativity and add *torsion* (1.12) which leads to the Riemann-Cartan spacetime and possible nonmetricity (1.16). In metric-affine gravity, spacetime is viewed as a connected real 4-dimensional manifold equipped with a Lorentzian metric and an affine connection. Unlike general relativity, in metric-affine gravity the metric and the connection are dynamic variables. The spacetime of metric-affine gravity reduces to the spacetime of the general relativity under the assumption that the torsion is zero and that the connection metric compatible. Metric-affine gravity is directly derived from the gauge theory of gravity where the linear connection plays the role of a gauge fields. The unknowns of metric-affine gravity are the 10 components of the metric tensor and the 64 connection coefficients. An advanced and comprehensive overview of this theory can be found in [13, 50, 51, 72, 73, 74, 75, 76, 77, 99, 100, 101].

In this thesis we are primarily concerned with the study of *quadratic metric-affine gravity* (QMAG), see Section 1.3. The mathematical model of this theory is that the Riemannian manifold is characterised by the principle of the action, which is a functional defined as an integral over the 4-dimensional manifold whose integrand is a purely quadratic form on curvature. The metric is still of Lorentzian signature and the unknown quantities are the components of the metric and the connection. Using variational calculus, we determine the stationary functions, i.e. the functions in which the change of the action is zero, which determines the metric and the connection. Mathematically, we consider the action defined by (1.1) where the Lagrangian is a quadratic form on curvature. Varying the action (1.1) with respect to the metric and the connection independently, we produce the system of Euler-Lagrange equation (1.2), (1.3). Our goal is to analyse the Riemannian and the non-Riemannian solutions of the system (1.2), (1.3), see Definition 1.3.3.

A large body of work is devoted to the analysis of Riemannian and non-Riemannian solutions of the Euler-Lagrange system (1.2), (1.3) in the case of quadratic Lagrangians, see e.g. [11, 30, 39, 50, 68, 70, 77, 82, 83, 87, 88, 89, 94, 99, 100, 101]. An important result in this field was given by Vassiliev [101], who solved the problem of existence and uniqueness of Riemannian solutions of the system (1.2), (1.3) in the most general case of the quadratic forms with 16 R^2 terms. Vassiliev showed that there are only three types of Riemannian solutions of the system (1.2), (1.3), see Section 4 and Section 5 of [101] and also presented the construction of "torsion waves" as one non-Riemannian solution of the system (1.2), (1.3). The same solution previously was obtained independently by Singh and Griffiths [89] and King and Vassiliev [53] for the Yang-Mills case (1.7). Torsion waves presented in [101] can be considered as non-Riemannian analogues of a pp-space, see Definition 2.1.1. Pasic and Vassiliev [77] presented a generalisation of classical pp-waves to spacetimes with torsion and constructed new class of non-Riemannian solutions in the most general case of quadratic form. A physical interpretation for that class of non-Riemannian solutions was given by Pasic and Barakovic [74], where it was suggested that a generalised pp-wave of parallel Ricci curvature represents a metric-affine model for the massless neutrino.

Motivated by the result of Singh [87] who presented a solution of the system (1.2), (1.3) with purely axial torsion (1.26) for the Yang-Mills action (1.7), we previously constructed a new non-Riemannian solution in [75], by generalising classical pp-waves to metric compatible spacetimes with purely axial torsion and proving that they are new vacuum solutions of the system (1.2), (1.3) for the Yang-Mills action (1.7). In this thesis, we aim to prove that these spacetimes are also a new solution of the system (1.2), (1.3) for the more general quadratic form with 11 R^2 terms (1.4). In the future it is our aim to see whether these spacetimes are also solutions of the system (1.2), (1.3) for the most general quadratic form with 16 R^2 terms (1.6). Also, in view of the construction of a new non-Riemannian solution for quadratic metric-affine gravity, the paper of Singh [88] is of particular interest to us, where a new solution of the field equations (1.2), (1.3) for the Yang-Mills action (1.7) with purely trace torsion (1.25) is presented. It would be interesting to see whether it is possible to generalise classical pp-waves to metric compatible spacetimes with purely trace torsion, as was similarly done in [75] and to see whether they are solutions of the system (1.2), (1.3). It is important to stress the fact that Sing [87], [88] did not use the most general quadratic form with 16 R^2 terms and the presented solutions can not be obtained using the double duality ansatz, see [62].

The analysis of pp-waves has a long and fruitful history. Pp-waves in the Yang-Mills type of quadratic metric-affine gravity with nontrivial torsion and with nonzero nonmetricity which do not belong to the triplet ansatz class [50] were analised by Obukhov [68], the motivation for which came from his earlier work [67]. The quadratic form considered by Obukhov is the most general with 16 R^2 terms. According to Obukhov, although Einstein's general relativity is supported by experiments on a macroscopic scale, the gravitational interaction on the microscopic level are not well understood. Gravity gauge models represent an alternative description of gravity in the micro cosmos. The study of the exact solutions of metric-affine gravity is essential for the understanding and development of physical aspects such as quantisation, physics of hadrons, studies of the early universe, etc. A construction and comparison of wave solutions in different models can explain the physical contents and relations between the microscopic and macroscopic gravitational theory.

An interesting analysis of plane-fronted gravitational and electromagnetic waves in metric-affine gravity with a cosmological constant in the triplet ansatz class was done by Garcia et al. [39]. As the authors emphasise, for restricted irreducible pieces of torsion and nonmetricity, there are similarities between the Einstein-Maxwell system and the vacuum metric-affine gravity field equations. In the same paper, the authors give a review of pp-waves and electromagnetic waves in the Einstein-Maxwell theory and present pp-waves and electromagnetic waves as solutions of metric-affine gravity with the cosmological constant in the triplet ansatz class. These waves carry curvature, nonmetricity, torsion and an electromagnetic field.

Gravitational waves as exact solutions of quadratic metric-affine gravity are also analised by Baykal [11]. Using the null coframe formalism, the author introduces a new family of impulsive gravitational wave solutions in four dimensions for a specific Lagrangian.

1.2 Gravitational and massless neutrino fields

The non-Riemannian solutions of metric-affine gravity considered in this thesis and in [72, 73, 74, 75, 76, 77] are presented in the form of generalised pp-waves of parallel Ricci curvature. As stated in [74], classical pp-waves with parallel Ricci curvature are the Riemannian representative within the class of solutions of metric-affine gravity, called generalised pp-waves of parallel Ricci curvature. Torsion and torsion generated curvature of generalised pp-waves can be considered as waves traveling at the speed of light and classical pp-waves with parallel Ricci curvature are only a “gravitational imprint” created by a wave of some massless matter field. Therefore, it is of interest to compare these with solutions of the classical Einstein-Weyl theory, which describes the interaction between gravitational fields and massless neutrino fields.

The massless Dirac operator describes a massless neutrino living in a compact space. In this thesis we perform the spectral analysis of the massless Dirac operator on a 3−dimensional manifold. An advanced review of the theory of the Dirac operator in general can be found in e.g. [60]. An interesting analysis of the Dirac equation on 4−dimensional manifolds without boundary was done in [33], where the authors gave a non geometrical representation of the massless Dirac equation. The eigenvalues of the massless Dirac operator represent the energy levels of the massless particle and we are interested in studying the spectrum of that operator, i.e. the set of all eigenvalues of the operator. To our knowledge, the first explicit calculation of the spectrum of the massless Dirac operator for the flat torus was done by Friedrich [35]. In the same paper, the dependence of the spectrum on the choice of the spin structure was also shown. The spectrum of the massless Dirac operator on the sphere S^n can also be explicitly calculated, see e.g. Trautman [93] and Bär [10]. It turns out that in these two cases the spectrum is symmetric about zero. However, the analysis of Atiyah et al. [3, 4, 5, 6] shows that for the oriented Riemann 3-manifold there is no physical reason for the spectrum to be symmetric. It would mean that in these two cases there is no difference between the properties of the massless neutrino and the massless antineutrino. Therefore, the objective of our study is to break the spectral symmetry of the massless Dirac operator on the unit 3−torus and the unit 3−sphere.

A very significant result in this field was given by Vassiliev et al. [24] who managed to break the spectral symmetry of the massless Dirac operator on the 3−torus considering the eigenvalue $\lambda = 0$. In [24] the authors use the perturbations of the Euclidean metric and derive an asymptotic formula for the eigenvalue zero, i.e. the eigenvalue of the massless Dirac operator with the smallest modulus. The authors show that it is possible to choose a perturbation of the Euclidean metric to shift the eigenvalue zero and to obtain spectral asymmetry. In the same paper, the authors analyse the eta invariant $\eta_H(0)$, see e.g. [3, 4, 5, 6], of a first-order self-adjoint elliptic $m \times m$ matrix classical pseudo-differential operator H as a measure of spectral asymmetry of the operator. In the case of a finite number of eigenvalues the eta invariant is an integer and it is the number of positive eigenvalues minus the number of negative eigenvalues. For the perturbed massless Dirac operator the limit formula for the eta invariant is derived.

Using a similar approach as in [24], in this thesis we aim to derive the asymptotic formulae for the next two eigenvalues considering unit 3−torus in the so-called axisymmetric case, see Section 3.5, i.e. for the eigenvalues $\lambda = \pm 1$.

In the future, it will be interesting to derive the asymptotic formulae for the eigenvalues of the massless Dirac operator acting on 3−sphere and see if and when it is possible to obtain the spectral asymmetry.

1.3 Quadratic metric-affine gravity

We consider spacetime to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. The 10 independent components of the symmetric metric tensor $g_{\mu\nu}$ and the 64 connection coefficients ${\Gamma^\lambda}_{\mu\nu}$ are the unknowns of our theory. We define the action as

$$S := \int q(R), \tag{1.1}$$

where q is $O(1,3)$ invariant quadratic form on curvature R (1.17). Note that the coefficients of this quadratic form depend only on the metric. Independent variation of the action (1.1) with respect to the metric g and the connection Γ produces the system of Euler-Lagrange equations and symbolically we write

$$\partial S/\partial g = 0, \tag{1.2}$$

$$\partial S/\partial \Gamma = 0. \tag{1.3}$$

The system (1.2), (1.3) is system of $10 + 64$ partial differential equations with $10 + 64$ real unknown components of metric and connection tensor. We use a purely quadratic Lagrangian because we are hoping to describe phenomena whose characteristic wavelength is sufficiently small and curvature sufficiently large, see [101]. The beginning of the development of this theory we find in the work of Hermann Weyl [103]. He stipulated that the most natural gravitational action should be quadratic in curvature and involve all possible invariant quadratic combinations of curvature. The quadratic curvature Lagrangians were also considered by many authors, see [9, 26, 52, 57, 58, 78, 90, 99, 100, 101].

Since the curvature has the 11 irreducible pieces, see Section 1.4.1, the quadratic form $q(R)$ can be represented as

$$q^{(11)}(R) := \sum_{i=1}^{11} c_i (R^{(i)}, R^{(i)})_{YM}, \tag{1.4}$$

for some real constants c_i, where $(\cdot,\cdot)_{YM}$ is Yang-Mills inner product

$$(R, Q)_{\mathrm{YM}} := R^{\kappa}{}_{\lambda\mu\nu} Q^{\lambda}{}_{\kappa}{}^{\mu\nu}. \tag{1.5}$$

This representation of the quadratic form, with 11 R^2 terms was considered by Vassiliev [99]. However, as Vassiliev also stated in [101], there are in fact sixteen ways of squaring the irreducible pieces to a scalar because some of the irreducible pieces are isomorphic. Respecting this argument, the quadratic form $q(R)$ can be represented as

$$\begin{aligned} q^{(16)}(R) &:= b_1 \mathcal{R}^2 + b_1^* \mathcal{R}_*^2 \\ &+ \sum_{l,m=1}^{3} b_{6lm} (\mathcal{A}^{(l)}, \mathcal{A}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm} (\mathcal{S}^{(l)}, \mathcal{S}^{(m)}) + \sum_{l,m=1}^{2} b_{9lm}^* (\mathcal{S}_*^{(l)}, \mathcal{S}_*^{(m)}) \\ &+ b_{10} (R^{(10)}, R^{(10)})_{\mathrm{YM}} + b_{30} (R^{(30)}, R^{(30)})_{\mathrm{YM}}, \end{aligned} \tag{1.6}$$

with some real constants b_1, b_1^*, $b_{6lm} = b_{6ml}$, $b_{9lm} = b_{9ml}$, $b_{9lm}^* = b_{9ml}^*$, b_{10}, b_{30}. This quadratic form (1.6) has 16 R^2 terms and the scalars $\mathcal{R}$, $\mathcal{R}_*$, and the tensors $\mathcal{A}^{(l)}$, $\mathcal{S}^{(l)}$, $\mathcal{S}_*^{(l)}$, $R^{(10)}$ and $R^{(30)}$ are defined in Section 1.4.1. The inner products in (1.6) are defined with (1.5) and

$$(R, Q) := R_{\mu\nu} Q^{\mu\nu}.$$

Remark 1.3.1. The action (1.1) is conformally invariant, i.e. it does not change if we perform a Weyl rescaling of the metric $g \to e^{2f} g,\ f : M \to \mathbb{R}$, without changing the connection Γ.

The development of this theory begins with the Yang-Mills theory. The Yang-Mills action for the affine connection is a special case of (1.1) with

$$q_{YM}(R) := R^{\kappa}{}_{\lambda\mu\nu} R^{\lambda}{}_{\kappa}{}^{\mu\nu}. \tag{1.7}$$

For the quadratic form (1.7), the so-called Yang-Mills equation (1.3) was first analised by Yang [106]. He specialized the equation (1.3) for the Levi-Civita connection and he got the equation

$$\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} = 0, \tag{1.8}$$

where by *Ric* we denote the Ricci curvature (1.18).

Remark 1.3.2. The Yang-Mills action can also be obtained from (1.4) by choosing that $c_1 = \ldots = c_{11} = 1$.

Depending on the type of the connection of spacetime, we consider two types of solutions of the system (1.2), (1.3), namely *Riemannian* and *non-Riemannian* solutions.

Definition 1.3.3. We call a spacetime $\{M, g, \Gamma\}$ Riemannian if the connection is Levi-Civita, i.e. $\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$, and non-Riemannian otherwise.

Remark 1.3.4. When we look for Riemannian solutions of the system (1.2), (1.3), we are still varying the action (1.1) independently with respect to the metric and with respect to the connection and only after we finish the variations we use the fact that connection is Levi-Civita.

The difference between the model with the quadratic form (1.4) and the model with the quadratic form (1.6) can bee seen if one considers the specialisation of the equation (1.3) to the Levi-Civita connection. For the eleven parameter action equation (1.3) reduces to equation (1.8) whereas for the sixteen parameter action (1.6) equation (1.3) reduces to

$$\nabla Ric = 0. \tag{1.9}$$

The equations (1.8) and (1.9) are different and equation (1.9) is much more restrictive.

An important class of Riemannian solutions are so-called *Einstein spaces.*

Definition 1.3.5. An Einstein space is a Riemannian spacetime with $Ric = \Lambda g$ where Ric is Ricci curvature (1.18) and Λ is some real constant.

Examining the equation (1.8), Yang concluded that Einstein spaces satisfy equation (1.8). It was shown later by many authors that Einstein spaces are satisfying the system (1.2), (1.3) for the quadratic form (1.7), see e.g. Yang [106] and Mielke [62]. We therefore refer to the special case (1.7) of the field equations (1.2), (1.3) as the *Yang-Mielke theory of gravity.* There are many works devoted to the study of the system (1.2), (1.3) in the special case (1.7) and one can get an idea of the historical development of the Yang-Mielke theory of gravity from [31, 32, 71, 79, 90, 92, 91, 105].

Vassiliev [101] solved the problem of existence and uniqueness of solutions of the system (1.2), (1.3) for the most general case of quadratic action with 16 R^2 terms. In the same paper, it was shown that Einstein spaces, pp-waves with parallel Ricci curvature (see Section 2.1) and Riemannian spacetimes which have zero scalar curvature and are locally a product of Einstein 2-manifolds are the *only* Riemannian solutions of the system (1.2), (1.3) for the most general quadratic form (1.6). Hence, all new solutions of the system (1.2), (1.3) that we can find are non-Riemannian solutions.

We are particularly interested in the analysis of a class of spacetimes called *pseudoinstantons*, which were introduced by Vassiliev in [99].

Definition 1.3.6. We call a spacetime $\{M, g, \Gamma\}$ a pseudoinstanton if the connection is metric-compatible and curvature is irreducible and simple.

The metric compatibility means that $\nabla g = 0$, where ∇ denotes the covariant derivative (1.15). Irreducibility of curvature means that only one of eleven irreducible pieces of curvature is nonzero and all others are identically equal to zero. Simplicity means that the given irreducible subspace that provides the non-zero piece of curvature is not isomorphic to any other irreducible subspace. Hence, there are only three types of pseudoinstantons:

- *scalar pseudoinstanton*, where only scalar curvature is not identically zero;
- *pseudoscalar pseudoinstanton*, where only pseudoscalar curvature is not identically zero;
- *Weyl pseudoinstanton*, where only Weyl curvature is not identically zero.

Pseudoinstantons are a very important class of spacetimes because they represent solutions of the field equations (1.2), (1.3) for the most general quadratic form (1.6), as was proved by Vassiliev [99]. For the construction of one non-Riemannian pseudoinstanton in Minkowski spacetime, see [101].

1.4 Notation and background

The notation in this thesis follows [53, 73, 74, 75, 76, 77, 99, 101]. We denote local coordinates by x^μ, $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector field as

$$\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu, \quad \nabla_\mu v_\lambda := \partial_\mu v_\lambda - \Gamma^\nu{}_{\mu\lambda} v_\nu, \tag{1.10}$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the connection coefficients. The Christoffel symbol is

$$\{\Gamma\}^\lambda{}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} := \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}). \tag{1.11}$$

We denote by $\{\nabla\}$ the covariant derivative with respect to the Levi-Civita connection, i.e.

$$\{\nabla\}_\mu v^\lambda := \partial_\mu v^\lambda + \{\Gamma\}^\lambda{}_{\mu\nu} v^\nu.$$

We define torsion as

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \tag{1.12}$$

and contortion as

$$K^\lambda{}_{\mu\nu} := \frac{1}{2} \left(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu \right). \tag{1.13}$$

Torsion can be expressed by contortion as

$$T^\lambda{}_{\mu\nu} = K^\lambda{}_{\mu\nu} - K^\lambda{}_{\nu\mu}.$$

The Levi-Civita connection $\{\Gamma\}$ and full connection Γ are related as

$$\Gamma^\lambda{}_{\mu\nu} = \{\Gamma\}^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu}. \tag{1.14}$$

We say that our connection Γ is *metric compatible* if $\nabla g \equiv 0$, i.e.

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\mu} - \Gamma^\kappa{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu} g_{\mu\kappa} = 0. \tag{1.15}$$

We define *nonmetricity* Q by

$$Q_{\mu\alpha\beta} := \nabla_\mu g_{\alpha\beta}. \tag{1.16}$$

The curvature tensor is defined in terms of the affine connection as

$$R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta} \Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta} \Gamma^\eta{}_{\mu\lambda}, \tag{1.17}$$

Ricci curvature as

$$Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}, \tag{1.18}$$

scalar curvature as

$$\mathcal{R} := Ric^{\kappa}{}_{\kappa} \tag{1.19}$$

and trace-free Ricci curvature as

$$\mathcal{R}ic := Ric - \frac{1}{4}\mathcal{R}g.$$

We denote Weyl curvature by $\mathcal{W}$. Weyl curvature is the irreducible piece of curvature defined by the conditions

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= R_{\mu\nu\kappa\lambda}, \\ \varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} &= 0, \\ Ric &= 0. \end{aligned} \tag{1.20}$$

Remark 1.4.1. The torsion and curvature can be written also in *anholonomic* notation which differs to the holonomic notation used in this thesis. The torsion is defined as

$$T^{\alpha} := D\vartheta^{\alpha} = \frac{1}{2}T_{ij}{}^{\alpha}dx^{i} \wedge dx^{j}$$

and the curvature as

$$R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha}{}_{\gamma} \wedge \Gamma^{\gamma\beta} = \frac{1}{2}R_{ij}{}^{\alpha\beta}dx^{i} \wedge dx^{j},$$

where ϑ^{α} is orthonormal frame and $\Gamma_{\alpha}{}^{\beta}$ is linear connection. For more about the anholonomic notation of curvature and torsion see Chapter 2 of [13].

We raise and lower tensor indices in the standard way, i.e. $g_{\alpha\beta}v^{\beta} = v_{\alpha}$, $g^{\alpha\beta}v_{\beta} = v^{\alpha}$. We define the action of the Hodge star on a rank q antisymmetric tensor as

$$(*Q)_{\mu_{q+1}\ldots\mu_4} := (q!)^{-1}\sqrt{|\det g|}\; Q^{\mu_1\ldots\mu_q}\varepsilon_{\mu_1\ldots\mu_4}\,, \tag{1.21}$$

where ε is the totally antisymmetric quantity and $\varepsilon_{0123} := +1$. When we apply the Hodge star to curvature we have a choice between acting either on the first or the second pair of indices, so we introduce two different operators: the left Hodge star

$$({}^{*}R)_{\kappa\lambda\mu\nu} := \frac{1}{2}\sqrt{|\det g|}\;\varepsilon^{\kappa'\lambda'}{}_{\kappa\lambda}\;R_{\kappa'\lambda'\mu\nu} \tag{1.22}$$

and the right Hodge star

$$(R^{*})_{\kappa\lambda\mu\nu} := \frac{1}{2}\sqrt{|\det g|}\;R_{\kappa\lambda\mu'\nu'}\,\varepsilon^{\mu'\nu'}{}_{\mu\nu}\,. \tag{1.23}$$

Given a scalar function $f : M \to R$ we write for brevity

$$\int f := \int f\sqrt{|\det g|}\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \quad \det g := \det(g_{\mu\nu}).$$

1.4.1 Irreducible pieces of torsion and curvature

In this section we provide the details of the irreducible pieces of torsion and curvature, where we mostly follow the exposition from [72, 99, 101].

Irreducible pieces of torsion

Under the local Lorentz group, the torsion T with its 24 independent components, can be irreducibly decomposed into three pieces. According to [51, 99], the irreducible pieces of torsion are

$$T^{(1)} = T - T^{(2)} - T^{(3)}, \tag{1.24}$$

$$T^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu, \tag{1.25}$$

$$T^{(3)} = *w, \tag{1.26}$$

where

$$v_\nu = \frac{1}{3}T^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{6}\sqrt{|\det g|}T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}. \tag{1.27}$$

The pieces $T^{(1)}, T^{(2)}$ i $T^{(3)}$ are called *tensor torsion* with its 16 independent components, *trace torsion* with its 4 independent components and *axial torsion* with its 4 independent components.

We define the action of the Hodge star on torsion as

$$(*T)_{\lambda\mu\nu} := \frac{1}{2}\sqrt{|\det g|}T_{\lambda\xi\eta}\,\varepsilon^{\xi\eta}{}_{\mu\nu}. \tag{1.28}$$

The Hodge star maps tensor torsion to tensor torsion, trace to axial, and axial to trace:

$$\begin{aligned}(*T)^{(1)} &= *(T^{(1)}),\\ (*T)^{(2)}{}_{\lambda\mu\nu} &= g_{\lambda\mu}w_\nu - g_{\lambda\nu}w_\mu,\\ (*T)^{(3)} &= -*v.\end{aligned} \tag{1.29}$$

The decomposition described above assumes torsion to be real and metric to be Lorentzian.

Remark 1.4.2. The Hodge star which appears in the RHS's of the formulae (1.26) and (1.29) is the standard Hodge star (1.21) and differs from the Hodge star on torsions (1.28).

Using formulae (1.12), (1.13) and (1.24)-(1.26), we obtain the irreducible decomposition of contortion

$$K^{(1)} = K - K^{(2)} - K^{(3)},$$
$$K^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\nu\mu}v_\lambda,$$
$$K^{(3)} = \frac{1}{2} * w,$$

where
$$v_\nu = \frac{1}{3}K^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{3}\sqrt{|\det g|}K^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}.$$

The irreducible pieces of torsion and contortion are related as[1]

$$T^{(1)}{}_{\kappa\lambda\mu} = K^{(1)}{}_{\lambda\kappa\mu},\ T^{(2)}{}_{\kappa\lambda\mu} = K^{(2)}{}_{\lambda\kappa\mu},\ T^{(3)}{}_{\kappa\lambda\mu} = 2K^{(3)}{}_{\kappa\lambda\mu}.$$

Irreducible pieces of curvature - version 1

In this section we present one decomposition of the curvature tensor generated by a general affine connection, where we follow the exposition from [99]. We denote by $\mathbf{R}$ the 96-dimensional vector space of real rank 4 tensors $R^\kappa{}_{\lambda\mu\nu}$. A curvature generated by a general affine connection has only one antisymmetry which satisfy the condition

$$R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}. \tag{1.30}$$

Let g be Lorentzian metric at the point $x \in M$ and let $O(1,3)$ be the corresponding full Lorentz group. The vector space $\mathbf{R}$ decomposes into a direct sum of 11 subspaces which are invariant and irreducible under the action of $O(1,3)$, see [51]. According to Vassiliev [99], we have the orthogonal decomposition $\mathbf{R} = \mathbf{R}^+ \oplus \mathbf{R}^-$ where

$$\mathbf{R}^\pm = \{R \in \mathbf{R} | R_{\kappa\lambda\mu\nu} = \pm R_{\lambda\kappa\mu\nu}\}$$

and $\dim \mathbf{R}^+ = 60$ and $\dim \mathbf{R}^- = 36$. The subspaces $\mathbf{R}^+$ and $\mathbf{R}^-$ decompose further into five and six irreducible subspaces, respectively. The vector space $\mathbf{R}^-$ is the vector space of curvatures generated by metric compatible connections and it can be written as $\mathbf{R}^- = \bigoplus\limits_{a,b=\pm} \mathbf{R}^-_{ab}$ where

$$\mathbf{R}^-_{ab} = \{R \in \mathbf{R}^- | R^T = aR,\ {}^*R^* = bR\},$$

[1] Note the order of indices.

where the map $R \to R^T$ defined with $(R^T)_{\kappa\lambda\mu\nu} := R_{\mu\nu\kappa\lambda}$ is endomorphism in $\mathbf{R}^-$. The map $R \to R^T$ is called *transposition* and the so-called *double duality map* $R \to {}^*R^*$ is defined with

$$ {}^*R^* := ({}^*R)^* = {}^*(R^*), $$

where $R \to {}^*R$ and $R \to R^*$ are left Hodge star (1.22) and right Hodge star (1.23) respectively.

Remark 1.4.3. The subspaces $\mathbf{R}^-_{++}$, $\mathbf{R}^-_{+-}$, $\mathbf{R}^-_{-+}$ and $\mathbf{R}^-_{--}$ are mutually orthogonal and their dimensions are $\dim \mathbf{R}^-_{++} = \dim \mathbf{R}^-_{-+} = 9$, $\dim \mathbf{R}^-_{+-} = 12$ and $\dim \mathbf{R}^-_{--} = 6$. The subspaces $\mathbf{R}^-_{++}$, $\mathbf{R}^-_{-+}$, $\mathbf{R}^-_{--}$ are irreducible and the only subspace which decomposes further is $\mathbf{R}^-_{+-}$ as

$$ \mathbf{R}^-_{+-} = \mathbf{R}_{\text{scalar}} \oplus \mathbf{R}_{\text{Weyl}} \oplus \mathbf{R}_{\text{pseudoscalar}}. $$

We now denote the subspaces $\mathbf{R}^-_{++}$, $\mathbf{R}_{\text{scalar}}$, $\mathbf{R}_{\text{Weyl}}$, $\mathbf{R}_{\text{pseudoscalar}}$, $\mathbf{R}^-_{-+}$, $\mathbf{R}^-_{--}$ by $\mathbf{R}^{(j)}$, $j = 1, \ldots, 6$, respectively.

The described decomposition assumes curvature to be real and metric to be Lorentzian. The explicit formulae for the six pieces of the curvature generated by metric compatible connections are

$$ R^{(1)} = \frac{1}{2}(g_{\kappa\mu}\overline{\mathcal{R}ic}_{\lambda\nu} - g_{\lambda\mu}\overline{\mathcal{R}ic}_{\kappa\nu} - g_{\kappa\nu}\overline{\mathcal{R}ic}_{\lambda\mu} + g_{\lambda\nu}\overline{\mathcal{R}ic}_{\kappa\mu}), \tag{1.31} $$

$$ R^{(2)} = \frac{1}{12}(g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu})\mathcal{R}, \tag{1.32} $$

$$ R^{(3)} = \bar{R} - R^{(1)} - R^{(2)} - R^{(4)}, \tag{1.33} $$

$$ R^{(4)} = -\frac{1}{24}\sqrt{|\det g|}\ \varepsilon_{\kappa\lambda\mu\nu}\breve{\mathcal{R}}, \tag{1.34} $$

$$ R^{(5)} = \hat{R} - R^{(6)}, \tag{1.35} $$

$$ R^{(6)} = \frac{1}{2}(g_{\kappa\mu}\widehat{Ric}_{\lambda\nu} - g_{\lambda\mu}\widehat{Ric}_{\kappa\nu} - g_{\kappa\nu}\widehat{Ric}_{\lambda\mu} + g_{\lambda\nu}\widehat{Ric}_{\kappa\mu}, \tag{1.36} $$

where

$$ \bar{R}_{\kappa\lambda\mu\nu} = \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} + R_{\mu\nu\kappa\lambda} - R_{\nu\mu\kappa\lambda}), $$

$$ \widehat{R}_{\kappa\lambda\mu\nu} = \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} - R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda}), $$

$$ \overline{Ric}_{\lambda\nu} = \bar{R}^{\kappa}{}_{\lambda\kappa\nu}, \quad \mathcal{R} = \overline{Ric}^{\lambda}{}_{\lambda} = R^{\kappa\lambda}{}_{\kappa\lambda}, \quad \overline{\mathcal{R}ic}_{\lambda\nu} = \overline{Ric}_{\lambda\nu} - \frac{1}{4}g_{\lambda\nu}\mathcal{R}, $$

$$ \widehat{Ric} = \widehat{R}^{\kappa}{}_{\lambda\kappa\nu}, \quad \breve{\mathcal{R}} = \sqrt{|\det g|}\varepsilon^{\kappa\lambda\mu\nu}\bar{R}_{\kappa\lambda\mu\nu} = \sqrt{|\det g|}\varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}. $$

Note that in the Riemannian case curvature has only three irreducible pieces $R^{(1)}, R^{(2)}$ and $R^{(3)}$.

Irreducible pieces of curvature-version 2

The vector space $\mathbf{R}$ can be decomposed in a different way, following the exposition from [72, 101]. The subspaces of the vector space $\mathbf{R}$ are:

- the two subspaces of dimension 1 noted by $\mathbf{R}^{(1)}$ and $\mathbf{R}_*^{(1)}$,
- the three subspaces of dimension 6 noted by $\mathbf{R}^{(6,l)}$, $(l = 1, 2, 3)$,
- the four subspaces of dimension 9 noted by $\mathbf{R}^{(9,l)}$ and $\mathbf{R}_*^{(9,l)}$, $(l = 1, 2)$,
- the one subspace of dimension 10 noted as $\mathbf{R}^{(10)}$, and
- the one subspace of dimension 30 noted as $\mathbf{R}^{(30)}$.

Two subspaces are said to be isomorphic if there is a linear bijection between them which commutes with the action of $O(1, 3)$. There are three groups of isomorphic subspaces, namely

$$\{\mathbf{R}^{(6,l)}, l = 1, 2, 3\}, \quad \{\mathbf{R}^{(9,l)}, l = 1, 2\}, \quad \{\mathbf{R}_*^{(9,l)}, l = 1, 2\}.$$

We provide the explicit formulae of the irreducible subspaces of dimension less than 10:

(a) the subspace $\mathbf{R}^{(1)}$ has the formula for the curvature

$$R_{\kappa\lambda\mu\nu} = a_1(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}, \tag{1.37}$$

(b) the subspace $\mathbf{R}_*^{(1)}$ has the formula for the curvature

$$(R^*)_{\kappa\lambda\mu\nu} = a_1^*(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})\mathcal{R}_*, \tag{1.38}$$

(c) the subspaces $\mathbf{R}^{(6,l)}$, $(l = 1, 2, 3)$ have the formula for the curvature

$$\begin{aligned} R_{\kappa\lambda\mu\nu} = {} & a_{6l1}(g_{\kappa\mu}\mathcal{A}^{(l)}{}_{\lambda\nu} - g_{\kappa\nu}\mathcal{A}^{(l)}{}_{\lambda\mu}) + a_{6l2}(g_{\lambda\mu}\mathcal{A}^{(l)}{}_{\kappa\nu} - g_{\lambda\nu}\mathcal{A}^{(l)}{}_{\kappa\mu}) \\ & + a_{6l3}g_{\kappa\lambda}\mathcal{A}^{(l)}{}_{\mu\nu}, \end{aligned} \tag{1.39}$$

(d) the subspaces $\mathbf{R}^{(9,l)}$, $(l = 1, 2)$ have the formula for the curvature

$$R_{\kappa\lambda\mu\nu} = a_{9l1}(g_{\kappa\mu}\mathcal{S}^{(l)}{}_{\lambda\nu} - g_{\kappa\nu}\mathcal{S}^{(l)}{}_{\lambda\mu}) + a_{9l2}(g_{\lambda\mu}\mathcal{S}^{(l)}{}_{\kappa\nu} - g_{\lambda\nu}\mathcal{S}^{(l)}{}_{\kappa\mu}), \tag{1.40}$$

(e) the subspaces $\mathbf{R}_*^{(9,l)}$, $(l = 1, 2)$ have the formula for the curvature

$$(R^*)_{\kappa\lambda\mu\nu} = a_{9l1}^*(g_{\kappa\mu}\mathcal{S}_*^{(l)}{}_{\lambda\nu} - g_{\kappa\nu}\mathcal{S}_*^{(l)}{}_{\lambda\mu}) + a_{9l2}^*(g_{\lambda\mu}\mathcal{S}_*^{(l)}{}_{\kappa\nu} - g_{\lambda\nu}\mathcal{S}_*^{(l)}{}_{\kappa\mu}), \tag{1.41}$$

where $a_1 = a_1^* = 1/12$ and

$$(a_{6lm}) = \begin{pmatrix} 5/12 & -1/12 & -1/6 \\ -1/12 & 5/12 & -1/6 \\ -1/12 & -1/12 & 1/3 \end{pmatrix}, \quad (a_{9lm}) = (a_{9lm}^*) = \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix}.$$

The scalars $\mathcal{R}, \mathcal{R}_*$ and the tensors $\mathcal{A}^{(l)}, \mathcal{S}^{(l)}, \mathcal{S}_*^{(l)}$ appearing in formulae (1.37)-(1.41) are expressed via the full curvature tensor R according to following formulae:

$$\begin{aligned}
&\mathcal{R} := R^{\kappa\lambda}{}_{\kappa\lambda}, \\
&Ric^{(1)}{}_{\mu\nu} := R^{\kappa}{}_{\mu\kappa\nu}, && Ric^{(2)}{}_{\mu\nu} := R_{\mu}{}^{\kappa}{}_{\kappa\nu}, \\
&\mathcal{R}ic^{(1)} := Ric^{(1)} - \frac{1}{4}\mathcal{R}g, && \mathcal{R}ic^{(2)} := Ric^{(2)} + \frac{1}{4}\mathcal{R}g, \\
&\mathcal{S}^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}{}_{\mu\nu} + \mathcal{R}ic^{(l)}{}_{\nu\mu}}{2}, && \mathcal{A}^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic^{(l)}{}_{\mu\nu} - \mathcal{R}ic^{(l)}{}_{\nu\mu}}{2}, \quad (l = 1, 2) \\
&\mathcal{A}^{(3)}{}_{\mu\nu} := R^{\kappa}{}_{\kappa\mu\nu},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}_* := (R^*)^{\kappa\lambda}{}_{\kappa\lambda}, \\
&Ric_*^{(1)}{}_{\mu\nu} := (R^*)^{\kappa}{}_{\mu\kappa\nu}, && Ric_*^{(2)}{}_{\mu\nu} := (R^*)_{\mu}{}^{\kappa}{}_{\kappa\nu}, \\
&\mathcal{R}ic_*^{(1)} := Ric_*^{(1)} - \frac{1}{4}\mathcal{R}_*g, && \mathcal{R}ic_*^{(2)} := Ric_*^{(2)} + \frac{1}{4}\mathcal{R}_*g, \\
&\mathcal{S}_*^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic_*^{(l)}{}_{\mu\nu} + \mathcal{R}ic_*^{(l)}{}_{\nu\mu}}{2}, && \mathcal{A}_*^{(l)}{}_{\mu\nu} := \frac{\mathcal{R}ic_*^{(l)}{}_{\mu\nu} - \mathcal{R}ic_*^{(l)}{}_{\nu\mu}}{2}, \quad (l = 1, 2) \\
&\mathcal{A}_*^{(3)}{}_{\mu\nu} := (R^*)^{\kappa}{}_{\kappa\mu\nu}.
\end{aligned}$$

Remark 1.4.4. Note that the tensor $Ric^{(1)}{}_{\mu\nu}$ and the Ricci tensor (1.18) are equivalent.

Remark 1.4.5. The tensors $\mathcal{A}_*^{(l)}$ does not appear in formulae (1.37)-(1.41). This is because the tensors $\mathcal{A}^{(l)}$ and $\mathcal{A}_*^{(l)}$ are not independent and the tensors $\mathcal{A}^{(l)}$ are linear combinations of the Hodge duals of the tensors $\mathcal{A}_*^{(l)}$, and vice versa, see [101]. Now we can write

$$\mathcal{A}_*^{(1)} = \alpha_1 \ *\mathcal{A}^{(1)} + \alpha_2 \ *\mathcal{A}^{(2)}, \tag{1.42}$$

$$\mathcal{A}_*^{(2)} = \beta_1 \ *\mathcal{A}^{(1)} + \beta_2 \ *\mathcal{A}^{(2)}, \tag{1.43}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are an arbitrary scalars.

The subspace $\mathbf{R}^{(10)}$ is the subspace of curvatures R such that all possible traces are equal to zero, i.e.

$$R^{\kappa}{}_{\lambda\kappa\nu} = (R^{*})^{\kappa}{}_{\lambda\kappa\nu} = 0, \quad R_{\mu}{}^{\kappa}{}_{\kappa\nu} = (R^{*})_{\mu}{}^{\kappa}{}_{\kappa\nu} = 0, \quad R^{\kappa}{}_{\kappa\mu\nu} = 0 \tag{1.44}$$

and $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$. The subspace $\mathbf{R}^{(30)}$ is the subspace of curvatures R such that it satisfies the relations (1.44) and $R_{\kappa\lambda\mu\nu} = R_{\lambda\kappa\mu\nu}$. Using the described subspaces, the space $\mathbf{R}$ has a following decomposition

$$\mathbf{R} = \mathbf{R}^{(1)} \oplus \mathbf{R}_{*}^{(1)} \oplus_{i=1}^{3} \mathbf{R}^{(6,l)} \oplus_{i=1}^{2} \mathbf{R}^{(9,l)} \oplus_{i=1}^{2} \mathbf{R}_{*}^{(9,l)} \oplus \mathbf{R}^{(10)} \oplus \mathbf{R}^{(30)}$$

and it means that arbitrary element $R \in \mathbf{R}$ can be uniquely written as

$$R = R^{(1)} + R_{*}^{(1)} + \sum_{l=1}^{3} R^{(6,l)} + \sum_{l=1}^{2} R^{(9,l)} + \sum_{l=1}^{2} R_{*}^{(9,l)} + R^{(10)} + R^{(30)},$$

where the R's in the RHS in the formulae (1.37)-(1.41).

Remark 1.4.6. We call the subspaces $\mathbf{R}^{(1)}$, $\mathbf{R}_{*}^{(1)}$, $\mathbf{R}^{(10)}$, $\mathbf{R}^{(30)}$ simple because they are not isomorphic to any other subspaces. Accordingly, we call the irreducible pieces $R^{(1)}$, $R_{*}^{(1)}$, $R^{(10)}$, $R^{(30)}$ simple.

1.4.2 On the Levi-Civita tensor

A very often used tensor in linear algebra, tensor analysis and differential geometry is so-called *Levi-Civita tensor.*

Definition 1.4.7. The Levi-Civita tensor is defined as

$$\epsilon_{\kappa\lambda\mu\nu} := \sqrt{|\det g|}\,\varepsilon_{\kappa\lambda\mu\nu}, \tag{1.45}$$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is totally antisymmetric quantity and $\varepsilon_{0123} = +1$.

Remark 1.4.8. In this thesis, we are mostly dealing with the Minkowski metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ and the pp-metric (2.1), whose determinant is equal to 1, and in those two cases the Levi-Civita tensor (1.45) is totally antisymmetric quantity $\varepsilon_{\kappa\lambda\mu\nu}$.

Lemma 1.4.9. *The covariant derivative of Levi-Civita tensor is zero, i.e.*

$$\nabla_{\xi}\epsilon_{\kappa\lambda\mu\nu} = 0. \tag{1.46}$$

Proof. Since the tensor $\epsilon_{\kappa\lambda\mu\nu}$ is by definition a totally antisymmetric tensor, it is enough to calculate calculate $\nabla_\xi \epsilon_{0123}$. Using the definition of the covariant derivative (1.10), we have that

$$\begin{aligned}\nabla_\xi \epsilon_{0123} &= \partial_\xi \epsilon_{0123} - \Gamma^\eta{}_{\xi 0}\epsilon_{\eta 123} - \Gamma^\eta{}_{\xi 1}\epsilon_{0\eta 23} - \Gamma^\eta{}_{\xi 2}\epsilon_{01\eta 3} - \Gamma^\eta{}_{\xi 3}\epsilon_{012\eta}\\ &= \partial_\xi \epsilon_{0123} - \Gamma^0{}_{\xi 0}\epsilon_{0123} - \Gamma^1{}_{\xi 1}\epsilon_{0123} - \Gamma^2{}_{\xi 2}\epsilon_{0123} - \Gamma^3{}_{\xi 3}\epsilon_{0123}\\ &= \partial_\xi \epsilon_{0123} - \sqrt{|\det g|}\Gamma^\eta{}_{\xi\eta}.\end{aligned}$$

The partial derivative is

$$\begin{aligned}\partial_\xi \epsilon_{0123} &= \partial_\xi(\sqrt{|\det g|})\varepsilon_{0123} + \sqrt{|\det g|}\partial_\xi(\varepsilon_{0123})\\ &= \frac{1}{2}\frac{1}{\sqrt{|\det g|}}\partial_\xi|\det g| = \sqrt{|\det g|}\{\Gamma\}^\eta{}_{\xi\eta}\end{aligned}$$

since $\{\Gamma\}^\eta{}_{\xi\eta} = \dfrac{\partial_\xi|\det g|}{2|\det g|}$. So, using formula (1.14), we get that

$$\nabla_\xi \epsilon_{0123} = \sqrt{|\det g|}\{\Gamma\}^\eta{}_{\xi\eta} - \sqrt{|\det g|}\Gamma^\eta{}_{\xi\eta} = -\sqrt{|\det g|}K^\eta{}_{\xi\eta}.$$

The contortion tensor K is antisymmetric in first and third index and since we have contraction in these two indices, it is equal to zero, and hence we get the equation (1.46). □

Remark 1.4.10. Direct calculations show that the totally antisymmetric quantity $\varepsilon_{\kappa\lambda\mu\nu}$ satisfies the following identities:

$$\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\kappa\lambda\mu\nu} = -4!, \tag{1.47}$$

$$\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\eta\lambda\mu\nu} = -3!\ \delta_\kappa{}^\eta, \tag{1.48}$$

$$\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\eta\xi\mu\nu} = -2!\ (\delta_\kappa{}^\eta\delta_\lambda{}^\xi - \delta_\kappa{}^\xi\delta_\lambda{}^\eta), \tag{1.49}$$

$$\begin{aligned}\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\eta\xi\zeta\nu} = -(&\delta_\kappa{}^\eta\delta_\lambda{}^\xi\delta_\mu{}^\zeta + \delta_\kappa{}^\xi\delta_\lambda{}^\zeta\delta_\mu{}^\eta + \delta_\kappa{}^\zeta\delta_\lambda{}^\eta\delta_\mu{}^\xi\\ &- \delta_\kappa{}^\eta\delta_\lambda{}^\zeta\delta_\mu{}^\xi - \delta_\kappa{}^\xi\delta_\lambda{}^\eta\delta_\mu{}^\zeta - \delta_\kappa{}^\zeta\delta_\lambda{}^\xi\delta_\mu{}^\eta).\end{aligned} \tag{1.50}$$

The Levi-Civita tensor is also appearing in the explicit formula (B.3) for the piece $R^{(5)}$. The relation between the Levi-Civita tensor (1.45) and the Weyl curvature is very interesting. It is known that the product of the Levi-Civita tensor and Weyl curvature with four contractions is equal to zero, see (1.20). It is of interest to see what happens in some other cases when contracting the Levi-Civita and Weyl tensors.

Lemma 1.4.11. *The Levi-Civita tensor is related to the Weyl tensor in the following way:*

$$\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\lambda\eta\xi} = 2\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\eta\lambda\xi}, \tag{1.51}$$

$$\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\lambda\mu\xi} = 0. \tag{1.52}$$

Proof. Using the well known Bianchi identity for the Weyl curvature

$$\mathcal{W}_{\kappa\lambda\eta\xi} + \mathcal{W}_{\kappa\eta\xi\lambda} + \mathcal{W}_{\kappa\xi\lambda\eta} = 0,$$

we have that

$$\epsilon^{\kappa\lambda\mu\nu}(\mathcal{W}_{\kappa\lambda\eta\xi} + \mathcal{W}_{\kappa\eta\xi\lambda} + \mathcal{W}_{\kappa\xi\lambda\eta}) = 0.$$

Renaming some indices and using the antisymmetry of $\epsilon_{\kappa\lambda\mu\nu}$ and the known properties of the Weyl tensor, we get that

$$\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\lambda\eta\xi} = -\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\eta\xi\lambda} - \epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\xi\lambda\eta} = 2\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\eta\lambda\xi},$$

which gives us the equation (1.51).

Contracting the indices μ and η in (1.51) and using the antisymmetry properties of the Weyl tensor we get $\epsilon^{\kappa\lambda\mu\nu}\mathcal{W}_{\kappa\lambda\mu\xi} = 0$, which is exactly equation (1.52). □

1.4.3 Spinor formalism

In this section we present the spinor formalism used in this thesis. We use the formalism introduced in [72, 74, 77].

Definition 1.4.12. The 'metric spinor' is defined as

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.53}$$

with the first index enumerating rows and the second enumerating columns.

We raise and lower spinor indices according to the formulae

$$\xi^a = \epsilon^{ab}\xi_b, \qquad \xi_a = \epsilon_{ab}\xi^b, \qquad \eta^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\eta_{\dot{b}}, \qquad \eta_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\eta^{\dot{b}}. \tag{1.54}$$

In this choice of spinor, the 'contravariant' and 'covariant' metric spinors are raised and lowered versions of each other, i.e. $\epsilon^{ab} = \epsilon^{ac}\epsilon_{cd}\epsilon^{bd}$ and $\epsilon_{ab} = \epsilon_{ac}\epsilon^{cd}\epsilon_{bd}$. Also, the spinor inner product is invariant under the operation of raising and lowering of indices, i.e. $(\epsilon_{ac}\xi^c)(\epsilon^{ad}\eta_d) = \xi^a\eta_a$. But, consecutive raising and lowering of a single spinor index leads to a change of sign, i.e. $\epsilon_{ab}\epsilon^{bc}\xi_c = -\xi_a$.

Definition 1.4.13. Let $\mathfrak{v}$ be the real vector space of Hermitian 2×2 matrices $\sigma_{a\dot{b}}$. Pauli matrices $\sigma^{\alpha}{}_{a\dot{b}}$, $\alpha = 0, 1, 2, 3$, are a basis in $\mathfrak{v}$ satisfying $\sigma^{\alpha}{}_{a\dot{b}}\sigma^{\beta c\dot{b}} + \sigma^{\beta}{}_{a\dot{b}}\sigma^{\alpha c\dot{b}} = 2g^{\alpha\beta}\delta_a{}^c$ where

$$\sigma^{\alpha a\dot{b}} := \epsilon^{ac}\sigma^{\alpha}{}_{c\dot{d}}\epsilon^{\dot{b}\dot{d}}. \tag{1.55}$$

Remark 1.4.14. At every point of the manifold M Pauli matrices are defined uniquely up to a Lorentz transformation.

Definition 1.4.15. The second order Pauli matrices are

$$\sigma_{\alpha\beta ac} := \frac{1}{2}\big(\sigma_{\alpha a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\beta c\dot{d}} - \sigma_{\beta a\dot{b}}\epsilon^{\dot{b}\dot{d}}\sigma_{\alpha c\dot{d}}\big), \tag{1.56}$$

where ϵ is the metric spinor (1.53).

These matrices are polarized, i.e.

$$*\sigma = \pm\mathrm{i}\,\sigma, \tag{1.57}$$

depending on the orientation of basic Pauli matrices $\sigma^\alpha{}_{a\dot{b}}$, $\alpha = 0, 1, 2, 3$.

Definition 1.4.16. We define the covariant derivatives of spinor fields as

$$\nabla_\mu \xi^a = \partial_\mu \xi^a + \Gamma^a{}_{\mu b}\xi^b, \qquad \nabla_\mu \xi_a = \partial_\mu \xi_a - \Gamma^b{}_{\mu a}\xi_b,$$
$$\nabla_\mu \eta^{\dot{a}} = \partial_\mu \eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}}\eta^{\dot{b}}, \qquad \nabla_\mu \eta_{\dot{a}} = \partial_\mu \eta_{\dot{a}} - \bar{\Gamma}^{\dot{b}}{}_{\mu\dot{a}}\eta_{\dot{b}},$$

where $\bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}} = \overline{\Gamma^a{}_{\mu b}}$.

The explicit formula for the spinor connection coefficients $\Gamma^a{}_{\mu b}$ can be derived from the following two conditions:

$$\nabla_\mu \epsilon^{ab} = 0, \quad \nabla_\mu j^\alpha = \sigma^\alpha{}_{a\dot{b}}\nabla_\mu \zeta^{a\dot{b}}, \tag{1.58}$$

where ζ is an arbitrary rank 2 mixed spinor field and $j^\alpha := \sigma^\alpha{}_{a\dot{b}}\zeta^{a\dot{b}}$ is the corresponding vector field. Conditions (1.58) give a system of linear algebraic equations for $\operatorname{Re}\Gamma^a{}_{\mu b}$, $\operatorname{Im}\Gamma^a{}_{\mu b}$ the unique solution of which is

$$\Gamma^a{}_{\mu b} = \frac{1}{4}\sigma_\alpha{}^{a\dot{c}}\left(\partial_\mu \sigma^\alpha{}_{b\dot{c}} + \Gamma^\alpha{}_{\mu\beta}\sigma^\beta{}_{b\dot{c}}\right). \tag{1.59}$$

1.5 Introducing the massless Dirac operator

The massless Dirac operator describes a single massless neutrino living in a compact universe and physically its eigenvalues are interpreted to be the energy levels of that massless particle. Using the massless Dirac operator, see Section 3.2, the massless Dirac equation can be obtained by varying the action (3.12) with respect to the spinor ξ. As we show in Section 2.4, the spinor field which determines the complexified curvature of the axial torsion waves introduced in Section 2.2.2 is an exact solution of the massless Dirac equation, see Lemma 2.4.2. We have the same situation in the case of purely tensor torsion waves introduced in Section 2.2.1. Hence, we have a connection between the generalised pp-waves with purely axial torsion and a massless neutrino field.

Remark 1.5.1. In this thesis, when we consider the solutions of the massless Dirac equation we work in the standard gravitational setting of four dimensions as opposed to the three dimensional setting when we perform spectral analysis of the massless Dirac operator.

Let M be a 3-dimensional connected compact oriented manifold equipped with a Riemannian metric g. The massless Dirac operator, see Definition 3.2.1, is the matrix operator

$$W = -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) \right).$$

To our knowledge, the eigenvalues of the massless Dirac operator can be explicitly calculated when we consider the unit torus $\mathbb{T}^3$ equipped with Euclidean metric and the unit sphere $\mathbb{S}^3$ equipped with the metric induced by the natural embedding of $\mathbb{S}^3$ in the Euclidean space $\mathbb{R}^4$. It turns out that in these cases the spectrum of the massless Dirac operator is symmetric. However, according to [3, 4, 5, 6], for a general oriented Riemannian 3-manifold (M, g) there is no physical reason for the spectrum of the massless Dirac operator to be symmetric. The spectral symmetry would mean that in these two examples, the massless neutrino and the massless antineutrino have the same properties. Hence, we are interested in a more detailed spectral analysis of the massless Dirac operator and creating spectral asymmetry.

Primarily, we consider the unit torus $\mathbb{T}^3$ equipped with Euclidean metric. In that case, the spectrum of the massless Dirac operator is as follows: zero is an eigenvalue of multiplicity two and for each $m \in \mathbb{Z}^3 \setminus \{0\}$ the eigenvalues are $\pm\|m\|$. Our goal is to break this spectral symmetry using the perturbations of Euclidean metric, see Section 3.4 and to derive the asymptotic formulae of the eigenvalues $\lambda = \pm 1$ in powers of the small perturbation parameter ϵ. For the eigenvalue $\lambda = 0$ of the massless Dirac operator the asymptotic formula was obtained in [24]. In the same paper the authors determined the conditions under which it is possible to obtain spectral asymmetry. Also, the authors give two explicit examples of perturbations of the Euclidean metric, for which the eigenvalues of the massless Dirac operator on half-densities (3.20) have been evaluated explicitly. One is an example of quadratic dependence and the second is an example of quartic dependence on parameter ϵ.

Similarly to the approach in [24], in this thesis we derive the asymptotic formulae for the eigenvalues ± 1 in the axisymmetric case, see Remark 3.5.2. We show that it is not possible to break the spectral asymmetry on the linear term, see Remark 3.5.9 and we also determine the perturbations of the Euclidean metric for which it is possible to obtain spectral asymmetry, see Remark 3.5.11.

Our aim in the near future is to obtain spectral asymmetry in the case of the unit sphere $\mathbb{S}^3$ equipped with the metric induced by the natural embedding of $\mathbb{S}^3$ in the Euclidean space $\mathbb{R}^4$.

1.6 Main results of the thesis

In this section we briefly explain the main results of the thesis. In order to construct new non-Riemannian solutions for the quadratic metric-affine gravity, we consider the generalisations of classical pp-waves to metric compatible spacetimes whose connection is not Levi-Civita. The whole of Chapter 2 is devoted to generalising pp-waves by adding torsion. Our new generalisation of pp-waves is introduced in Section 2.2.2, where we present the spacetime whose torsion is purely axial and where we list its main properties. We deal with a particular choice of local coordinates where the pp-metric can be written as

$$ds^2 = 2\,dx^0\,dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1,x^2,x^3)\,(dx^3)^2$$

in some local coordinates (x^0,x^1,x^2,x^3). Then we define generalised pp-waves with axial torsion as metric compatible spacetime with the pp-metric and torsion

$$T := *A,$$

where A is a real vector field defined by $A = k(\varphi)\,l$, where $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function and l is a parallel null light-like vector $l^\mu = (1,0,0,0)$. Torsion defined in such a way is purely axial, see Lemma 2.2.9, and in our local coordinates it can be written as

$$T_{\kappa\mu\nu} = k(x^3) l^\alpha \varepsilon_{\alpha\kappa\mu\nu}.$$

The explicit formula for the curvature in our local coordinates is

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l\wedge\partial)_{\alpha\beta}\,(l\wedge\partial)_{\gamma\delta}f + \sum_{i,j=1}^{2} r_{ij}(l\wedge m_i)_{\alpha\beta}(l\wedge m_j)_{\gamma\delta},$$

where $r_{11} = r_{22} = \frac{1}{4}(k(x^3))^2$, $r_{12} = -r_{21} = \pm\frac{1}{2}k'(x^3)$, $m^\mu = (0,1,\mp\mathrm{i},0)$, $m_1 = \mathrm{Re}(m)$ and $m_2 = \mathrm{Im}(m)$.

One of the main results of this thesis is the following

Theorem 1.6.1. *Generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of* (1.2), (1.3) *in the case* (1.7).

In order to prove that generalised pp-waves solve the system of equations (1.2), (1.3) in the case (1.7), we first explicitly write down these equations in the form (2.47) and (2.48), which is already well known, see e.g. [53]. Then using (2.8), special local coordinates (2.1), (2.5), as well as the formulae for curvature (2.40), (2.41), torsion (2.35) and torsion generated curvature (2.43) we prove that equations (2.47) and (2.48) are satisfied.

Another main result is the following

Theorem 1.6.2. *Generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of* (1.2), (1.3) *in the case* (1.4).

In order to prove Theorem 1.6.2, we write down the field equations (1.2), (1.3) explicitly under the certain assumptions, see Section 2.3.1. Then we show that the field equations are satisfied by substituting the explicit formulae for the irreducible pieces of torsion and curvature of generalised pp-waves, see Section 2.3.4.

The generalised pp-waves of parallel Ricci curvature with purely axial torsion have their particular physical interpretation, as we show in Section 2.4. The spinor field ξ, which completely determines the complexified curvature of generalised pp-waves, satisfies the massless Dirac equation. The axial torsion is the irreducible piece of torsion which is usually used when one models the massless neutrino, see [20], or the electron, see [16], by means of Cosserat elasticity. In Section 2.4 we compare the generalised pp-waves with purely axial torsion, as new vacuum solutions for quadratic metric-affine gravity, to the solutions of the Einstein-Weyl theory. We conclude that generalised pp-waves with purely axial torsion of parallel Ricci curvature are very similar to pp-type solutions of the Einstein-Weyl model and we propose that generalised pp-waves with purely axial torsion of parallel Ricci curvature represent a metric-affine model for the massless neutrino.

Further, in Chapter 3, we are interested in a more mathematical approach to the analysis of the massless Dirac operator on a $3-$dimensional manifold. As we stated in Section 1.5, physically interpreted, the massless Dirac operator describes a single massless neutrino. We consider the unit torus $\mathbb{T}^3$ equipped with the Euclidean metric where the spectrum is calculated explicitly. It turns out that in this case the spectrum is symmetric about zero. Our concern about the spectral symmetry arises from the fact that for a general oriented Riemannian 3-manifold (M, g) there is no physical reason for the spectrum of the massless Dirac operator to be symmetric. Hence, our aim is to create spectral asymmetry. For creating spectral asymmetry we can choose one of the two different approaches: we can consider $3-$manifolds with flat metric but nontrivial topology as in [81] or the trivial topology with a perturbed metric as in [24]. We have decided for the latter.

Our third main result is the following

Theorem 1.6.3. *Under certain perturbations of the Euclidean metric, we can obtain spectral asymmetry of the massless Dirac operator (in the axisymmetric case) on the unit* $3-$*torus.*

In order to prove Theorem 1.6.3, we consider the perturbed Euclidean metric

$$g_{\alpha\beta}(x;\epsilon) = \delta_{\alpha\beta} + \epsilon h_{\alpha\beta}(x) + \frac{\epsilon^2}{4} k_{\alpha\beta}(x) + O(\epsilon^3),$$

where ϵ is a positive small parameter. Using the perturbation theory developed for the massless Dirac operator, see Section 3.4, we explicitly derive the asymptotic formulae for the eigenvalues $\lambda = \pm 1$ of the massless Dirac operator in the axisymmetric case (3.57). In terms of perturbation matrices $h_{\alpha\beta}$ and $k_{\alpha\beta}$ the asymptotic formulae are given in Theorem 3.5.7. Analysing these asymptotic formulae, we see that under certain perturbations of the Euclidean metric, it is possible to obtain spectral asymmetry of the massless Dirac operator in the axisymmetric case, see Remark 3.5.9 and Remark 3.5.11.

1.7 Structure of the thesis

This thesis has the following structure:

- Chapter 2 deals with new solutions for quadratic metric-affine gravity and is divided into several sections. In Section 2.1 we present the classical pp-waves and we list the main properties of these spacetimes. In Section 2.2 we are dealing with generalisations of classical pp-waves, extending classical pp-waves to metric compatible spacetimes with torsion. We introduce generalised pp-waves with purely tensor torsion and generalised pp-waves with purely axial torsion and list the main properties of these spacetimes. In Section 2.3 we use generalised pp-waves with purely axial torsion described in Section 2.2 to present a class of new solutions for quadratic metric-affine gravity. In Section 2.4 we give the mathematical and physical significance of our new solutions. We compare our new solutions for quadratic metric-affine gravity to the pp-type solutions of Einstein-Weyl theory which describes the interaction of gravitational and massless neutrino fields.

- Chapter 3 deals with the spectral analysis of the massless Dirac operator on a $3-$manifold. In Section 3.1 we list some general properties of an elliptic self-adjoint first-order differential operator. In Section 3.2

we present the massless Dirac operator and we list its main properties. The Section 3.3 deals with the spectrum of the massless Dirac operator with special emphasis on studying the spectral function and the counting function of the massless Dirac operator. In Section 3.4 we develop the perturbation theory for the massless Dirac operator used through this thesis. Section 3.5 deals with spectral asymmetry of the massless Dirac operator which is the main objective of Chapter 3. In this section we present our new results for obtaining spectral asymmetry of the massless Dirac operator.

- Appendices provide the auxiliary mathematical facts and the calculations used to obtain the results in this thesis. Appendix A provides the detailed derivation of the Einstein and Yang-Mills equations. Appendix B gives the derivation of the Bianchi identity for the curvature which is used to obtain the explicit representations of the field equations in Section 2.3. In Appendix C we provide the explicit variations of certain quadratic forms on curvature with respect to the metric which are used to get the explicit representations of the field equations. Finally, Appendix D contains the detailed calculations of the coefficients in the asymptotic expansion of the eigenvalues ± 1 of the massless Dirac operator.

Chapter 2

New Solutions for Quadratic Metric-Affine Gravity

2.1 Classical pp-waves

Pp-waves are a very important family of exact solutions of Einstein's field equations and are very well known spacetimes in general relativity. They are introduced long time ago by Brinkmann [14] in 1923, and after rediscovered by several authors. Pp-wave metrics are interpreted as the metrics representing the gravitational waves by Peres [80]. As stated Baykal [12], the pp-wave metrics can be regarded as a far-field description of an isolated astrophysical source radiating gravitational waves. Griffiths [47] and Kramer et al. [55] gave an explanation that the abbreviation 'pp' stands for 'plane-fronted gravitational waves with parallel rays'. It was recently discovered by Vassiliev [101] that pp-waves of parallel Ricci curvature are solutions of the system (1.2), (1.3). For more information on pp-waves and pp-wave type solutions of metric-affine gravity, see e.g. [1, 8, 13, 14, 39, 47, 55, 67, 68, 73, 77, 80, 100, 101]. In this chapter, classical pp-waves will be used for the construction of the new solutions for quadratic metric-affine gravity. We mostly follow the exposition of [74, 75].

Definition 2.1.1. A *pp-wave* is a Riemannian spacetime whose metric can be written locally in the form

$$ds^2 = 2\,dx^0\,dx^3 - (dx^1)^2 - (dx^2)^2 + f(x^1, x^2, x^3)\,(dx^3)^2 \tag{2.1}$$

in some local coordinates (x^0, x^1, x^2, x^3).

The advantage of Definition 2.1.1 is that it gives an explicit formula for the metric of a pp-wave but its disadvantage is that it relies to the specific coordinate system. We also have the coordinate-free definition of the pp-wave spacetime.

Definition 2.1.2. A *pp-wave* is a Riemannian spacetime which admits a nonvanishing parallel spinor field.

Remark 2.1.3. The term *parallel* means that covariant derivative of the spinor field is zero, see Definition 1.4.16.

It is known, see [2, 15], that Definition 2.1.1 and Definition 2.1.2 are equivalent. The nonvanishing parallel spinor field appearing in Definition 2.1.2 of pp-waves will be denoted throughout this chapter by

$$\chi = \chi^a, \tag{2.2}$$

and we assume this spinor field to be *fixed*. Let define the vector l as

$$l^\alpha := \sigma^\alpha{}_{a\dot{b}}\, \chi^a \bar{\chi}^{\dot{b}} \tag{2.3}$$

where σ^α are Pauli matrices, see Definition 1.4.13 and Section 1.4.3. Clearly, l is a nonvanishing parallel real null vector field. Using the vector l, we define the real scalar function

$$\varphi : M \to \mathbb{R}, \qquad \varphi(x) := \int l \;\cdot dx\,. \tag{2.4}$$

This function is called the *phase*.

Definition 2.1.4. The complex vector field v is transversal if $l_\alpha v^\alpha = 0$.

Definition 2.1.5. The complex vector field v is plane wave if $v^\alpha \nabla_\alpha v^\beta = 0$ for arbitrary transversal vector field v.

Remark 2.1.6. Clearly, the vector l is a transversal and a plane wave.

The choice of local coordinates in which the pp-metric has the form (2.1) is not unique. Our choice of those coordinates is such in which

$$\chi^a = (1, 0), \qquad l^\mu = (1, 0, 0, 0), \qquad m^\mu = (0, 1, \mp\mathrm{i}, 0). \tag{2.5}$$

With the choice (2.5), the phase function (2.4) explicitly reads $\varphi(x) = x^3 +$ const. Using the second order Pauli matrices (1.56) and the spinor field (2.2) we can define the complex 2-form

$$F_{\alpha\beta} := \sigma_{\alpha\beta ab}\, \chi^a \chi^b. \tag{2.6}$$

Also, it can be written as the wedge product

$$F = l \wedge m, \tag{2.7}$$

where m is a complex vector field satisfying

$$m_\alpha m^\alpha = l_\alpha m^\alpha = l_\alpha \overline{m}^\alpha = 0, \; m_\alpha \bar{m}^\alpha = -2. \tag{2.8}$$

If we denote by $m_1 = \mathrm{Re}(m)$ and $m_2 = \mathrm{Im}(m)$, then from (2.8) we also have the following relations

$$l_\alpha l^\alpha = m_{1\alpha} m_2{}^\alpha = l_\alpha m_1{}^\alpha = l_\alpha m_2{}^\alpha = 0, \; m_{1\alpha} m_1{}^\alpha = m_{2\alpha} m_2{}^\alpha = -1. \tag{2.9}$$

Remark 2.1.7. Clearly, F is a nonvanishing parallel complex 2-form. Applying the Hodge star operator (1.21) in (2.7) we get that $*F = \pm \mathrm{i} F$. Also $\det F = 0$ for the choice (2.5).

For the metric (2.1), we get the explicit formula for the curvature of the classical pp-wave

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l \wedge \partial)_{\alpha\beta} \, (l \wedge \partial)_{\gamma\delta} f, \tag{2.10}$$

where $(l \wedge \partial)_{\alpha\beta} := l_\alpha \partial_\beta - \partial_\alpha l_\beta$. The curvature tensor R is linear in f. Formula (2.10) can be written in invariant form

$$R = -\frac{1}{2}(l \wedge \nabla) \otimes (l \wedge \nabla) f, \tag{2.11}$$

where $l \wedge \nabla := l \otimes \nabla - \nabla \otimes l$. The curvature of a pp-wave has only two irreducible pieces, see Section 1.4.1, namely (symmetric) trace-free Ricci and Weyl. Ricci curvature (1.18) is proportional to $l \otimes l$ whereas Weyl curvature is a linear combination of $\mathrm{Re}\,((l \wedge m) \otimes (l \wedge m))$ and $\mathrm{Im}\,((l \wedge m) \otimes (l \wedge m))$. In our special local coordinates (2.1), (2.5), we can express Ricci and Weyl curvature as

$$\begin{aligned} Ric_{\mu\nu} &= \frac{1}{2}(f_{11} + f_{22}) \, l_\mu l_\nu, & (2.12) \\ \mathcal{W}_{\kappa\lambda\mu\nu} &= \sum_{j,k=1}^{2} w_{jk} (l \wedge m_j) \otimes (l \wedge m_k), & (2.13) \end{aligned}$$

where $f_{\alpha\beta} := \partial_\alpha \partial_\beta f$ and w_{jk} are real scalars given by

$$w_{11} = \frac{1}{4}(-f_{11} + f_{22}), \; w_{12} = \pm\frac{1}{2} f_{12}, \; w_{22} = -w_{11}, \; w_{21} = w_{12}.$$

The Cotton tensor, see [40], of classical pp-waves is given by

$$C_{\lambda\mu\nu} = \nabla_\lambda Ric_{\mu\nu} - \nabla_\mu Ric_{\lambda\nu}. \tag{2.14}$$

In the theory of conformal spaces the main geometrical objects to be analised are the Weyl and the Cotton tensors, see [40]. It is well known that for conformally flat spaces the Weyl tensor has to vanish and consequently the Cotton tensor has to vanish too. The Cotton tensor is only conformally invariant in three dimensions.

Note that if the Ricci curvature is parallel the Cotton tensor (2.14) vanishes.

2.1.1 Pauli matrices for pp-waves

We choose the Pauli matrices for the pp-metric (2.1.1) as

$$\begin{aligned}
\sigma^0{}_{a\dot b} &= \begin{pmatrix} 1 & 0 \\ 0 & -f \end{pmatrix}, & \sigma^1{}_{a\dot b} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2{}_{a\dot b} &= \begin{pmatrix} 0 & \mp\mathrm{i} \\ \pm\mathrm{i} & 0 \end{pmatrix}, & \sigma^3{}_{a\dot b} &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.
\end{aligned} \tag{2.15}$$

Our two choices of Pauli matrices differ by orientation. When dealing with a classical pp-wave the choice of orientation of Pauli matrices does not really matter, but for a generalised pp-wave it is convenient to choose orientation of Pauli matrices in agreement with the sign in (2.19) as this simplifies the resulting formulae.

Remark 2.1.8. In the case $f = 0$, formulae (2.15) do not turn into the standard Minkowski space Pauli matrices, since we write the metric in the form (2.1). This is a matter of convenience in calculations.

The second order Pauli matrices $\sigma^{\alpha\beta}{}_{ab}$ (1.56) for the pp-metric (2.1) are also antisymmetric over the tensor indices, so we only give the independent non-zero components. The explicit formulae for the second order Pauli matrices for the pp-metric (2.1) are

$$\begin{aligned}
\sigma^{01}{}_{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, & \sigma^{02}{}_{ab} &= \begin{pmatrix} \mp\mathrm{i} & 0 \\ 0 & \pm\mathrm{i}f \end{pmatrix}, & \sigma^{03}{}_{ab} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^{12}{}_{ab} &= \begin{pmatrix} 0 & \mp\mathrm{i} \\ \mp\mathrm{i} & 0 \end{pmatrix}, & \sigma^{13}{}_{ab} &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, & \sigma^{23}{}_{ab} &= \begin{pmatrix} 0 & 0 \\ 0 & \pm 2\mathrm{i} \end{pmatrix}.
\end{aligned} \tag{2.16}$$

2.2 Generalising pp-waves

As we previously stated, the classical pp-waves of parallel Ricci curvature are solutions of the system (1.2), (1.3). Since the classical pp-waves are Riemannian spacetimes with zero torsion, we use these spacetimes in order to construct new solutions of the quadratic metric-affine gravity with torsion. In this section we present the generalisation of the classical pp-wave to the spacetimes whose connection is not necessarily Levi-Civita, but in a very particular way: we will still use the pp-metric (2.1) and introduce explicitly given torsion. We present two types of such spacetimes: *generalised pp-waves with purely tensor torsion* and *generalised pp-waves with purely axial torsion*. The former were introduced and analised in [72, 74, 77], while the latter were first introduced and analised to a certain degree in [75].

2.2.1 Generalised pp-waves with purely tensor torsion

This generalisation of the classical pp-waves is done by Pasic and Vassiliev [77]. In the same paper it was shown that the generalised pp-waves with purely tensor torsion of parallel Ricci curvature are solutions of the system (1.2), (1.3) in the general case with 16 R^2 quadratic form (1.6). The physical description of that new solutions is given by Pasic and Barakovic [74]. Here we present a review of these results.

Definition 2.2.1. A *generalised pp-wave with purely tensor torsion* is a metric compatible spacetime with pp-metric (2.1) and torsion

$$T := \frac{1}{2}\mathrm{Re}(A \otimes dA), \tag{2.17}$$

where A is a vector field of the form

$$A = h(\varphi)\, m \,+\, k(\varphi)\, l, \tag{2.18}$$

which is a plane wave solution of the polarized Maxwell equation

$$*dA = \pm \mathrm{i}\, dA. \tag{2.19}$$

The vector fields l and m appearing in (2.18) are defined in Section 2.1, $h, k : \mathbb{R} \to \mathbb{C}$ are arbitrary functions, and φ is the phase (2.4).

Remark 2.2.2. We denote by $\{\nabla\}$ the covariant derivative with respect to the Levi-Civita connection which should be distinguished from the full covariant derivative ∇ incorporating torsion.

These spacetimes have an explicit formulae for the curvature and the torsion. This remarkable property is not a trivial fact. The curvature of a generalised pp-wave is

$$R = -\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4}\mathrm{Re}\left((h^2)''\,(l \wedge m) \otimes (l \wedge m)\right). \tag{2.20}$$

Remark 2.2.3. The curvatures generated by the Levi-Civita connection and torsion simply add up (compare formulae (2.11) and (2.20)).

The torsion of a generalised pp-wave is

$$T = \mathrm{Re}\left((a\ l + b\ m) \otimes (l \wedge m)\right), \tag{2.21}$$

where

$$a := \frac{1}{2}h'(\varphi)\ k(\varphi), \quad b := \frac{1}{2}h'(\varphi)\ h(\varphi). \tag{2.22}$$

Torsion can be written down even more explicitly in the following form

$$T = \sum_{j,k=1}^{2} t_{jk}\, m_j \otimes (l \wedge m_k) + \sum_{j=1}^{2} t_j\, l \otimes (l \wedge m_j), \tag{2.23}$$

where

$$t_{11} = -t_{22} = \frac{1}{2}\mathrm{Re}(b),\ t_{12} = t_{21} = -\frac{1}{2}\mathrm{Im}(b),\ t_1 = \frac{1}{2}\mathrm{Re}(a),\ t_2 = -\frac{1}{2}\mathrm{Im}(a),$$

where a and b are the functions (2.22).

Remark 2.2.4. The torsion (2.17) of a generalised pp-wave is purely tensor. For the proof see Lemma 2 of [74].

The spinor filed (2.2) introduced in Section 2.1 satisfies $\{\nabla\}\chi = 0$. As was shown in [74], for the generalised pp-waves with purely tensor torsion we also have that $\nabla\chi = 0$. It means that the generalised pp-wave with purely tensor torsion and the underlying classical pp-wave admit the same nonvanishing parallel spinor field. Also, the generalised pp-wave and the underlying classical pp-wave admit the same nonvanishing parallel real null vector field l and the same nonvanishing parallel complex 2-form (2.6), (2.7).

We list below the main properties of generalised pp-waves with purely tensor torsion:

(a) The second term in the explicit formula for the curvature (2.20) of the generalised pp-wave with purely tensor torsion is purely Weyl.

(b) The generalised pp-waves with purely tensor torsion have the same non-zero irreducible pieces of curvature as classical pp-waves, namely symmetric trace-free Ricci and Weyl. Using special local coordinates (2.1), (2.5), these can be expressed explicitly as

$$Ric_{\mu\nu} = \frac{1}{2}(f_{11} + f_{22})\, l_\mu l_\nu, \tag{2.24}$$

$$\mathcal{W}_{\kappa\lambda\mu\nu} = \sum_{j,k=1}^{2} w_{jk}(l \wedge m_j) \otimes (l \wedge m_k), \tag{2.25}$$

where w_{jk} are real scalars given by

$$\begin{aligned} w_{11} &= \frac{1}{4}[-f_{11} + f_{22} + \mathrm{Re}((h^2)'')], \qquad & w_{22} &= -w_{11} \\ w_{12} &= \pm\frac{1}{2}f_{12} - \frac{1}{4}\mathrm{Im}((h^2)''), \qquad & w_{21} &= w_{12}. \end{aligned}$$

The formulae (2.24) and (2.25) are very similar to the formulae (2.12) and (2.13). The only difference is in the coefficients w_{ij}.

(c) The Ricci curvature (2.24) of a generalised pp-wave with purely tensor torsion is completely determined by the pp-metric (2.1).

(d) The curvature of a generalised pp-wave with purely tensor torsion (2.20) has all the usual symmetries of curvature in the Riemannian case, that is

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \tag{2.26}$$

$$\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \tag{2.27}$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}, \tag{2.28}$$

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}. \tag{2.29}$$

Of course, (2.29) is true for any curvature whereas (2.28) is a consequence of metric compatibility. Also, (2.28) follows from (2.26) and (2.29).

(e) The second term in the vector field A (2.18) does not affect curvature (2.20) and it only affects torsion (2.17).

(f) The Ricci curvature of a generalised pp-wave with purely tensor torsion (2.24) is parallel if and only if

$$f_{11} + f_{22} = C, \tag{2.30}$$

where C is an arbitrary real constant.

(g) The Ricci curvature of a generalised pp-wave with purely tensor torsion (2.24) is zero if and only if

$$f_{11} + f_{22} = 0 \tag{2.31}$$

and the Weyl curvature is zero if and only if

$$f_{11} - f_{22} = \mathrm{Re}\left((h^2)''\right), \qquad f_{12} = \pm\frac{1}{2}\mathrm{Im}\left((h^2)''\right). \tag{2.32}$$

Here we use special local coordinates (2.1), (2.5).

(h) The curvature of a generalised pp-wave with purely tensor torsion (2.20) is zero if and only if we have both (2.31) and (2.32).

2.2.2 Generalised pp-waves with purely axial torsion

In this section we present the generalisation of the classical pp-wave to the spacetimes whose torsion is purely axial. This generalisation of the classical pp-wave spacetimes was introduced by Pasic and Barakovic [75]. In the same paper it was shown that these spacetimes are solutions of the system (1.2), (1.3) for the Yang-Mills action (1.7). In this chapter we will show that generalised pp-waves with purely axial torsion are solutions of the system (1.2), (1.3) for the case quadratic form (1.4) with 11 R^2 terms.

Definition 2.2.5. A generalised pp-wave with purely axial torsion is a metric compatible spacetime with pp-metric and torsion

$$T := *A \tag{2.33}$$

where A is a real vector field defined by $A = k(\varphi)\, l$, where $k : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary real function of the phase (2.4) for the vector l (2.5).

Remark 2.2.6. In our special local coordinates (2.1), (2.5), torsion (2.33) can be written as

$$T_{\kappa\mu\nu} = k(x^3) l^\alpha \varepsilon_{\alpha\kappa\mu\nu}, \tag{2.34}$$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is totally antisymmetric quantity and $\varepsilon_{0123} = +1$.

Remark 2.2.7. The real vector field A is a plane wave solution of the polarized Maxwell equation $*dA = \pm\mathrm{i}\; dA$. This is not surprising as in our special local coordinates the vector field A is the gradient of a scalar function.

Remark 2.2.8. It has been suggested that one can interpret the axial component of torsion as the Hodge dual of the electromagnetic vector potential, see [53, 98].

Lemma 2.2.9. *The torsion (2.33) of a generalised pp-wave is purely axial.*

Proof. In our special local coordinates (2.1), (2.5), using formulae (1.27) and (2.34), we have that

$$w_\nu = \frac{1}{6}\sqrt{|\det g|}T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu} = \frac{1}{6}k(x^3)l_\alpha\varepsilon^{\alpha\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu}.$$

Hence, using formula (1.48), we get that

$$\begin{aligned}(T^{(3)})_{\alpha\beta\gamma} = (*w)_{\alpha\beta\gamma} &= \frac{1}{6}k(x^3)l_\alpha\varepsilon^{\alpha\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu}{}^{\eta}\varepsilon_{\eta\alpha\beta\gamma}\\ &= -\frac{1}{6}k(x^3)l_\xi(-3!)g^{\xi\eta}\varepsilon_{\eta\alpha\beta\gamma} = k(x^3)l^\xi\varepsilon_{\xi\alpha\beta\gamma} = T_{\alpha\beta\gamma}.\end{aligned}$$

□

Lemma 2.2.10. *Using our special local coordinates (2.1), (2.5), we can express torsion as*

$$T = \mp\frac{\mathrm{i}}{2}k(x^3)\ l\wedge m\wedge\overline{m}, \tag{2.35}$$

where the $\mp$ sign is chosen to correspond to the sign in (2.5).

Proof. We will prove that using our special local coordinates (2.1), (2.5) we have that

$$l_\mu\varepsilon^{\mu\alpha\beta\gamma} = \mp\frac{\mathrm{i}}{2}(l\wedge m\wedge\overline{m})^{\alpha\beta\gamma}. \tag{2.36}$$

By definition we have that

$$\begin{aligned}(l\wedge m\wedge\overline{m})^{\alpha\beta\gamma} &= l^\alpha m^\beta\overline{m}^\gamma + l^\beta m^\gamma\overline{m}^\alpha + l^\gamma m^\alpha\overline{m}^\beta\\ &\quad - l^\alpha m^\gamma\overline{m}^\beta - l^\beta m^\alpha\overline{m}^\gamma - l^\gamma m^\beta\overline{m}^\alpha.\end{aligned} \tag{2.37}$$

Using special local coordinates (2.1), (2.5), we conclude that if any of the α,β or γ is equal to 3, then the wedge product (2.37) is zero and the quantity $l_\mu\varepsilon^{\mu\alpha\beta\gamma} = l_3\varepsilon^{3\alpha\beta\gamma}$ is also zero since ε is totally antisymmetric. Hence (2.36) holds.

Let then $\alpha,\beta,\gamma\neq 3$. By definition, tensors $(l\wedge m\wedge\overline{m})^{\alpha\beta\gamma}$ and $l_\mu\varepsilon^{\mu\alpha\beta\gamma}$ are totally antisymmetric, hence the only non-zero terms in both quantities appear when α,β,γ are a permutation of $0,1,2$. It is therefore enough to calculate the only independent non-zero term, i.e.

$$(l\wedge m\wedge\overline{m})^{012} = l^0m^1\overline{m}^2 - l^0m^2\overline{m}^1 = \pm 2\mathrm{i}.$$

Now we have that $\mp\frac{\mathrm{i}}{2}(l\wedge m\wedge\overline{m})^{012} = 1$ and $l_\mu\varepsilon^{\mu012} = l_3\varepsilon_{3012} = 1$. The other cases are shown analogously. Hence we have the formula (2.36). Combining formulae (2.34) and (2.36), we get formula (2.35). □

Clearly, from the equation (2.35) we have that we can express contortion as

$$K = \mp \frac{\mathrm{i}}{4} k(x^3)\ l \wedge m \wedge \overline{m}. \tag{2.38}$$

Also, since we have that

$$l \wedge m \wedge \overline{m} = l \wedge (m_1 + im_2) \wedge (m_1 - im_2) = -2\mathrm{i}\,(l \wedge m_1 \wedge m_2),$$

where $m_1 = \mathrm{Re}(m)$ and $m_2 = \mathrm{Im}(m)$, we get an equivalent formula for the torsion

$$T = \mp k(x^3)\ l \wedge m_1 \wedge m_2 \tag{2.39}$$

in our special local coordinates (2.1), (2.5).

Remark 2.2.11. Our torsion completely corresponds to Singh's axial torsion from [87, 88]. Put $m = -\frac{1}{2}k(x^3)$ in formula (16) of [87] or put $n = 0, m = -\frac{1}{2}k(x^3)$ in formula (20) of [88].

Lemma 2.2.12. *The connection of a generalised pp-wave with purely axial torsion is metric compatible.*

Proof. Since

$$\Gamma^\kappa{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + K^\kappa{}_{\mu\nu} = \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + \frac{1}{2} T^\kappa{}_{\mu\nu},$$

we get that

$$\nabla_\mu g_{\alpha\beta} = \{\nabla\}_\mu g_{\alpha\beta} - K^\eta{}_{\mu\alpha} g_{\eta\beta} - K^\eta{}_{\mu\beta} g_{\alpha\eta}$$

and $\{\nabla\}_\mu g_{\alpha\beta} = 0$ as classical pp-waves are metric compatible. However, since our torsion is purely axial, we get that

$$\nabla_\mu g_{\alpha\beta} = -K_{\beta\mu\alpha} - K_{\alpha\mu\beta} = K_{\alpha\mu\beta} - K_{\alpha\mu\beta} = 0,$$

i.e. we have metric-compatibility. □

Remark 2.2.13. Note that the spinor field (2.2) is no longer parallel in generalised pp-waves with purely axial torsion, as was the case with generalised pp-waves with purely tensor torsion. Moreover, using local coordinates (2.1), (2.5), the covariant derivative of this spinor field is $\nabla \chi = \begin{pmatrix} 2i \\ 0 \end{pmatrix} \neq 0$.

The remarkable property of these spacetimes is that we have the explicit formula for the curvature which is

$$\begin{aligned} R = &-\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f + \frac{1}{4}(k(x^3))^2\ \mathrm{Re}\,(\,(l \wedge m) \otimes (l \wedge \overline{m})) \\ &\mp \frac{1}{2} k'(x^3)\ \mathrm{Im}\,(\,(l \wedge m) \otimes (l \wedge \overline{m}))\,. \end{aligned} \tag{2.40}$$

This can be equivalently written down as

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{2}(l \wedge \partial)_{\alpha\beta}\,(l \wedge \partial)_{\gamma\delta} f + \sum_{i,j=1}^{2} r_{ij}(l \wedge m_i)_{\alpha\beta}(l \wedge m_j)_{\gamma\delta}, \tag{2.41}$$

where $r_{11} = r_{22} = \frac{1}{4}(k(x^3))^2$, $r_{12} = -r_{21} = \pm\frac{1}{2}k'(x^3)$, and $m_1 = \mathrm{Re}(m)$, $m_2 = \mathrm{Im}(m)$. It is a highly nontrivial fact that the torsion generated curvature, i.e.

$$R_T{}^{\kappa}{}_{\lambda\mu\nu} = \partial_\mu K^\kappa{}_{\nu\lambda} - \partial_\nu K^\kappa{}_{\mu\lambda} + K^\kappa{}_{\mu\eta}K^\eta{}_{\nu\lambda} - K^\kappa{}_{\nu\eta}K^\eta{}_{\mu\lambda}, \tag{2.42}$$

which is equal to

$$\begin{aligned} R_T = {} & \frac{1}{4}(k(x^3))^2 \ \mathrm{Re}\,((l \wedge m) \otimes (l \wedge \overline{m})) \\ & \mp \frac{1}{2}k'(x^3) \ \mathrm{Im}\,((l \wedge m) \otimes (l \wedge \overline{m})) \end{aligned} \tag{2.43}$$

and the Riemannian curvature (2.11) simply add up to produce formula (2.40).

Remark 2.2.14. Note that the above property of curvature of the pp-waves with purely axial torsion corresponds to the similar property that pp-waves with purely tensor torsion also poses.

We know that the Riemannian part of curvature has two irreducible pieces of curvature, namely Weyl and (symmetric) trace-free Ricci. It turns out that the torsion also generates Ricci curvature and it reads

$$Ric_{\mu\nu} = \frac{1}{2}\left(f_{11} + f_{22} - (k(x^3))^2\right) l_\mu l_\nu \tag{2.44}$$

and

$$Ric_{*\mu\nu} = -k'(x^3) l_\mu l_\nu. \tag{2.45}$$

Scalar curvature is then clearly zero by the properties of the vector l. The curvature of generalised pp-waves with purely axial torsion (2.40) has only three irreducible pieces, namely $R^{(1)}$ (1.31), $R^{(3)}$ (1.33) and $R^{(5)}$ (1.35). The irreducible piece $R^{(1)}$ partly comes from the classical pp-waves and partly from torsion generate curvature. The irreducible piece $R^{(3)}$ entirely arises from the classical pp-wave and the irreducible piece $R^{(5)}$ from the torsion generated curvature (2.43). In our special local coordinates (2.1), (2.5), the explicit formula for the irreducible piece $R^{(1)}$ is given by

$$R^{(1)}{}_{\kappa\lambda\mu\nu} = \frac{1}{4}\,(f_{11}+f_{22})\,(g_{\kappa\mu}l_\lambda l_\nu - g_{\lambda\mu}l_\kappa l_\nu + g_{\lambda\nu}l_\kappa l_\mu - g_{\kappa\nu}l_\lambda l_\mu)$$
$$+\frac{1}{4}k^2\ \mathrm{Re}\,(\,(l\wedge m)_{\kappa\lambda}(l\wedge\overline{m})_{\mu\nu})$$

and the explicit formula for the Weyl curvature $R^{(3)}$ is given by the earlier formula (2.13). The explicit formula for the irreducible piece $R^{(5)}$ is given by

$$R^{(5)}{}_{\kappa\lambda\mu\nu} = \pm\frac{1}{2}\ k'(x^3)(\varepsilon^\eta{}_{\kappa\mu\nu}l_\lambda l_\eta - \varepsilon^\eta{}_{\lambda\mu\nu}l_\kappa l_\eta). \tag{2.46}$$

For the detailed derivation of the formula (2.46) see Appendix B.1.

Examination of the formula (2.40) for the curvature of a generalised pp-wave with purely axial torsion reveals the following properties:

(a) The curvatures generated by the Levi-Civita connection (2.11) and torsion (2.43) simply add up to produce the formula (2.40).

(b) The curvature of a generalised pp-wave has the symmetries

$$\varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} = 0,$$
$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu},$$
$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu},$$

but we do not have the symmetry $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\lambda\kappa}$ as was the case with generalised pp-waves with purely tensor torsion.

(c) For an arbitrary purely axial torsion, the Weyl curvature of the resulting torsion generated curvature (2.43) is zero.

(d) Ricci is parallel if $f_{11}+f_{22} = (k(x^3))^2 + C$, in which case $Ric = \Lambda\ l\otimes l$, for some constant Λ.

(e) The Ricci curvature (2.44) is zero if Poisson's equation $f_{11}+f_{22} = (k(x^3))^2$ is satisfied.

2.3 New solutions for quadratic metric-affine gravity

In this section we prove that the generalised pp-waves with purely axial torsion presented in Section 2.2.2 are the solutions of the system (1.2), (1.3) for the Yang-Mills action (1.7) and also in the case of the 11 R^2 term quadratic form (1.4).

First, we will prove that the generalised pp-waves with purely axial torsion are solutions of the system (1.2), (1.3) for the Yang-Mills action (1.7).

The main result is the following

Theorem 2.3.1. *Generalised pp-waves with purely axial torsion of parallel* $\{Ric\}$ *are solutions of* (1.2), (1.3) *in the special case* (1.7)*.*

Remark 2.3.2. Note that by $\{Ric\}$ we denote the Ricci curvature generated by the Levi-Civita connection only. The condition $\{\nabla\}\{Ric\} = 0$ implies that $f_{11} + f_{22} = C$. Note that the result also holds if Ric is assumed to be parallel.

Remark 2.3.3. In the special case (1.7), we call equation (1.3) the *Yang-Mills equation for the affine connection*, i.e.

$$\partial_\nu R^{\mu\nu} + [\Gamma_\nu, R^{\mu\nu}] = 0, \tag{2.47}$$

where $[\Gamma_\nu, R^{\mu\nu}]^\kappa{}_\lambda = \Gamma^\kappa{}_{\nu\eta} R^\eta{}_\lambda{}^{\mu\nu} - \Gamma^\eta{}_{\nu\lambda} R^\kappa{}_\eta{}^{\mu\nu}$. We call the equation (1.2) in the special case (1.7) the *complementary Yang-Mills equation*, i.e.

$$H - \frac{1}{4}(\mathrm{tr}\ H)g = 0, \tag{2.48}$$

where $H = H_\nu{}^\rho := R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_\kappa{}^{\mu\rho}$. Equivalently, equation (2.48) can written down as

$$R^\kappa{}_{\lambda\nu}{}^\alpha R^\lambda{}_\kappa{}^{\nu\beta} - \frac{1}{4} g^{\alpha\beta} R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_\kappa{}^{\mu\nu} = 0.$$

For the explicit derivations of the equations (2.47) and (2.48) see Appendix A.2.

Proof *of Theorem 2.3.1.* Since we know, see e.g. [74, 77, 100], that classical pp-waves of parallel Ricci curvature are solutions of (1.2), (1.3) in the special case (1.7), it is enough to show the result for the torsion generated part of curvature (2.43). In proving that generalised pp-waves solve the equations (2.47) and (2.48), we will use equations (2.8), special local coordinates (2.1), (2.5), as well as the formulae for curvature (2.40), (2.41), torsion (2.35) and torsion generated curvature (2.43). To show that equation (2.47) is satisfied, we only need to show that

$$\partial_\nu R_T{}^{\mu\nu} + [K_\nu, R_T{}^{\mu\nu}] = 0,$$

since the curvature is the sum of the Riemannian curvature (2.11) and the torsion generated curvature (2.43), the connection the sum of the Christoffel symbol and contortion and since classical pp-waves solve the Yang-Mills equation. Now, since $F = l \wedge m$, in special local coordinates (2.1), (2.5), the antisymmetric tensor F (2.6) only has two non-zero independent components, namely

$$F^{01} = 1, \quad F^{02} = \mp\mathrm{i}.$$

Since the function $k(\varphi)$ is the function of x^3 in special local coordinates, using the formula for curvature (2.43) we directly get that $\partial_\nu {R_T}^{\mu\nu} = 0$. Using the explicit formula for torsion (2.35), the fact that it is purely axial, which implies that $T = 2K$, the explicit formula for torsion generated curvature (2.43) and special local coordinates (2.1), (2.5), we get that the only nonzero term of ${\Gamma^\kappa}_{\nu\eta} {R^\eta}_\lambda{}^{\mu\nu}$ is for $\kappa = 0, \lambda = 3, \mu = 0$, i.e. $-\frac{1}{2}k \cdot k'$. However, the only nonzero term of ${\Gamma^\eta}_{\nu\lambda} {R^\kappa}_\eta{}^{\mu\nu}$ is also when $\kappa = 0, \lambda = 3, \mu = 0$, i.e. $-\frac{1}{2}k \cdot k'$, so these two terms cancel out. Hence, the equation (2.47) is satisfied. Checking that all the terms in equation (2.48) are zero is a straightforward, using the formulae for curvature (2.40), (2.41), special local coordinates (2.1), (2.5) and equation (2.8). $\square$

Now, we will prove that the generalised pp-waves with purely axial torsion are solutions of the system (1.2), (1.3) for the quadratic form (1.4). The main result of this chapter is the following

Theorem 2.3.4. *Generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of* (1.2), (1.3) *in the case* (1.4).

Remark 2.3.5. The condition of parallel Ricci curvature can be exchanged with the condition $\{\nabla\}\{Ric\} = 0$, see Remark 2.3.13.

In order to prove Theorem 2.3.4 we need to first explicitly write down our system of equations (1.2), (1.3).

2.3.1 Explicit representation of the field equations

In this section we explicitly write down the system of equations (1.2), (1.3) for the quadratic form (1.4) under the following assumptions:

(i) Our spacetime is metric compatible.

(ii) Torsion is purely axial.

(iii) Ricci curvature (1.18) is symmetric.

(iv) Scalar curvature $\mathcal{R}$ and pseudoscalar curvature $\mathcal{R}_*$ are zero.

Remark 2.3.6. Note that pp-waves with purely axial torsion automatically satisfy the above assumptions (i) – (iv).

Remark 2.3.7. Note that the above assumptions $(i)-(iv)$ are only applied *after* the variations have been performed.

Remark 2.3.8. In the general case, the curvature has the symmetry $R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}$ and the symmetry $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$ is the consequence of the metric compatibility. In the derivation of the explicit form of the system of equations (1.2), (1.3), *we will not use* the property that $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}$, since the curvature of the generalised pp-waves with axial torsion does not possess this property.

Remark 2.3.9. Note that the symmetry of Ric implies the symmetry of Ric_*. Indeed, under the assumption that Ric is symmetric consequently the tensors $A^{(l)}$ from Section 1.4.1 are all equal to zero. Equations (1.42), (1.43) imply that the tensors $\mathcal{A}_*^{(i)}$ are also equal to zero, which implies that Ric_* is also symmetric.

The main result of this section is the following.

Theorem 2.3.10. *Under the above assumptions* $(i)-(iv)$ *the field equations* (1.2), (1.3) *in the special case* (1.4), *can be written down as*

$$0 = 2d_1 W^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} + d_2 \epsilon^{\eta\nu\alpha\beta} Ric_{\kappa\nu} {Ric_*}^{\kappa}{}_{\eta} - d_3 \epsilon^{\kappa\lambda\xi\alpha} {W_{\kappa\lambda\mu}}^{\beta} {Ric_*}^{\mu}{}_{\xi}, \tag{2.49}$$

$$\begin{aligned}
0 = {} & d_1 \left\{ \nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + T_{\mu\eta\lambda} {Ric_\kappa}^{\eta} + T_{\mu\kappa\eta} {Ric_\lambda}^{\eta} \right\} \\
& - d_4 \left\{ (g_{\kappa\mu} \mathcal{W}^{\xi\zeta}{}_{\lambda\eta} - g_{\lambda\mu} \mathcal{W}^{\xi\zeta}{}_{\kappa\eta}) T^{\eta}{}_{\xi\zeta} + (g_{\kappa\mu} \epsilon^{\vartheta\zeta}{}_{\eta\lambda} - g_{\lambda\mu} \epsilon^{\vartheta\zeta}{}_{\eta\kappa}) T^{\eta}{}_{\xi\zeta} {Ric_*}^{\xi}{}_{\vartheta} \right\} \\
& + c_5 \left\{ \epsilon^{\eta\xi}{}_{\kappa\mu} \nabla_\xi Ric_{*\lambda\eta} - \epsilon^{\eta\xi}{}_{\lambda\mu} \nabla_\xi Ric_{*\kappa\eta} + \tfrac{1}{2} ({\epsilon_\kappa}^{\eta\xi\zeta} Ric_{*\lambda\eta} - {\epsilon_\lambda}^{\eta\xi\zeta} Ric_{*\kappa\eta}) T_{\mu\zeta\xi} \right\} \\
& - c_3 \left\{ 2T^{\eta}{}_{\lambda\xi} \mathcal{W}^{\xi}{}_{\mu\kappa\eta} + 2T^{\eta}{}_{\xi\kappa} \mathcal{W}^{\xi}{}_{\mu\lambda\eta} + T_{\mu\xi\eta} {\mathcal{W}_{\kappa\lambda}}^{\eta\xi} + \epsilon^{\vartheta}{}_{\mu\eta\lambda} T^{\eta}{}_{\xi\kappa} {Ric_*}^{\xi}{}_{\vartheta} \right. \\
& \qquad - \epsilon^{\vartheta}{}_{\mu\kappa\eta} T^{\eta}{}_{\xi\lambda} {Ric_*}^{\xi}{}_{\vartheta} - \epsilon^{\vartheta\xi}{}_{\eta\kappa} T^{\eta}{}_{\xi\lambda} Ric_{*\mu\vartheta} + \epsilon^{\vartheta\xi}{}_{\eta\lambda} T^{\eta}{}_{\xi\kappa} Ric_{*\mu\vartheta} \\
& \qquad \left. + \epsilon^{\vartheta}{}_{\mu\lambda\kappa} \nabla_\xi {Ric_*}^{\xi}{}_{\vartheta} - \epsilon^{\vartheta\xi}{}_{\lambda\kappa} \nabla_\xi Ric_{*\mu\vartheta} \right\},
\end{aligned} \tag{2.50}$$

where c_1, c_3, c_5 *are the coefficients of the quadratic form* (1.4) *and* $d_1 = c_1 + c_3$, $d_2 = c_1 - c_5$, $d_3 = c_3 + c_5$, $d_4 = \frac{1}{2}(c_1 - c_3)$.

Remark 2.3.11. Since the action (1.1) is conformally invariant, the system (2.49), (2.50) is actually the system of $9 + 64$ equations, i.e. the equation (2.49) has 9 independent equations, not 10.

Note that the equations (2.49), (2.50) are obtained by varying the action (1.1) for the quadratic form (2.52) independently with respect to the metric and with respect to the connection and without any a priori assumptions on the curvature or torsion. Only *after* we have finished these variations will we use the assumptions $(i)-(iv)$ and explicit formulae for the pieces of curvature in order to obtain the equations (2.49), (2.50).

Remark 2.3.12. Assumption (i) implies that the piece of curvature $R^{(6)}$ (1.36) is zero and assumption (iv) clearly implies that the $R^{(2)}$ (1.32) and $R^{(4)}$ (1.34) pieces of curvature are zero. Hence, under the above assumptions, the curvature (1.17) has only *three* nonzero irreducible pieces, namely $R^{(1)}$, $R^{(3)}$ and $R^{(5)}$. It can therefore be represented as

$$\begin{aligned} R_{\kappa\lambda\mu\nu} = &\frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} + g_{\lambda\nu}Ric_{\kappa\mu} - g_{\kappa\nu}Ric_{\lambda\mu}) + \mathcal{W}_{\kappa\lambda\mu\nu} \\ &+ \frac{1}{2}(-\epsilon^{\eta}{}_{\lambda\mu\nu}Ric_{*\kappa\eta} + \epsilon^{\eta}{}_{\kappa\mu\nu}Ric_{*\lambda\eta}). \end{aligned} \tag{2.51}$$

Now we will prove Theorem 2.3.10. Since the curvature of the generalised pp-waves with axial torsion (2.40) has only three irreducible pieces $R^{(1)}$ (1.31), $R^{(3)}$ (1.33) and $R^{(5)}$ (1.35), we consider the quadratic form (1.4) written as

$$q(R) = c_1(R^{(1)}, R^{(1)}) + c_3(R^{(3)}, R^{(3)}) + c_5(R^{(5)}, R^{(5)}) + \dots \tag{2.52}$$

where by ... we denote the terms of the quadratic form that do not contribute δS when we vary using the above assumptions (i)–(iv).

2.3.2 Variation with respect to metric

Using the formula (1.31) for the piece $R^{(1)}$ of curvature and the formula for the Yang-Mills inner product (1.5), we get that

$$(R^{(1)}, R^{(1)})_{YM} = R^{(1)\kappa}{}_{\lambda\mu\nu}R^{(1)\lambda}{}_{\kappa}{}^{\mu\nu} = -2\overline{Ric}_{\lambda\nu}\overline{Ric}^{\lambda\nu} + \frac{1}{2}\mathcal{R}^2.$$

Since the variation of the scalar curvature with respect to the metric is zero, we get that

$$\begin{aligned} (R^{(1)}, R^{(1)})_{YM} &= -2\overline{Ric}_{\lambda\nu}\overline{Ric}^{\lambda\nu} \\ &= -\frac{1}{2}Ric_{\lambda\nu}Ric^{\lambda\nu} - \frac{1}{2}Ric^{(2)}{}_{\lambda\nu}Ric^{(2)\lambda\nu} + Ric_{\lambda\nu}Ric^{(2)\lambda\nu}. \end{aligned}$$

Using the results from Appendix C we get that

$$\begin{aligned} &\frac{\delta}{\delta g}\int (R^{(1)}, R^{(1)})_{YM} \\ &\quad = \int (\delta g_{\alpha\beta})\left(2Ric^{\alpha}{}_{\nu}Ric^{\beta\nu} - 2R^{\kappa\beta\alpha\nu}Ric_{\kappa\nu} - Ric_{\lambda\nu}Ric^{\lambda\nu}g^{\alpha\beta}\right). \end{aligned} \tag{2.53}$$

Substituting the explicit formula for the curvature

$$
\begin{aligned}
R^{\kappa\lambda\mu\nu} &= R^{(1)\kappa\lambda\mu\nu} + R^{(3)\kappa\lambda\mu\nu} + R^{(5)\kappa\lambda\mu\nu} \\
&= \frac{1}{2}\left(g^{\kappa\mu}Ric^{\lambda\nu} - g^{\lambda\mu}Ric^{\kappa\nu} - g^{\kappa\nu}Ric^{\lambda\mu} + g^{\lambda\nu}Ric^{\kappa\mu}\right) \\
&+ W^{\kappa\lambda\mu\nu} + \frac{1}{2}\left(\epsilon^{\eta\kappa\mu\nu}Ric_*{}^{\lambda}{}_{\eta} - \epsilon^{\eta\lambda\mu\nu}Ric_*{}^{\kappa}{}_{\eta}\right)
\end{aligned}
\tag{2.54}
$$

into (2.53) we get

$$
\begin{aligned}
&\frac{\delta}{\delta g}\int (R^{(1)}, R^{(1)})_{YM} \\
&\quad = \int (\delta g_{\alpha\beta})\left(-2W^{\kappa\beta\alpha\nu}Ric_{\kappa\nu} + \epsilon^{\eta\beta\alpha\nu}Ric_*{}^{\kappa}{}_{\eta}Ric_{\kappa\nu}\right).
\end{aligned}
\tag{2.55}
$$

Since the piece $R^{(6)}$ of curvature is zero, according to formula (1.35), the piece $R^{(5)}$ of curvature is

$$
R^{(5)}_{\kappa\lambda\mu\nu} = \frac{1}{4}\left(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} - R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda}\right).
$$

Hence

$$
\begin{aligned}
&\frac{\delta}{\delta g}\int (R^{(5)}, R^{(5)})_{YM} = \\
&= \frac{1}{16}\frac{\delta}{\delta g}\int \left(R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu} - R^{\kappa}{}_{\lambda\mu\nu}R_{\kappa}{}^{\lambda\mu\nu} - R^{\kappa}{}_{\lambda\mu\nu}R^{\mu\nu\lambda}{}_{\kappa} + R^{\kappa}{}_{\lambda\mu\nu}R^{\nu\mu\lambda}{}_{\kappa}\right) \\
&- \frac{1}{16}\frac{\delta}{\delta g}\int \left(R_{\lambda}{}^{\kappa}{}_{\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu} - R_{\lambda}{}^{\kappa}{}_{\mu\nu}R_{\kappa}{}^{\lambda\mu\nu} - R_{\lambda}{}^{\kappa}{}_{\mu\nu}R^{\mu\nu\lambda}{}_{\kappa} + R_{\lambda}{}^{\kappa}{}_{\mu\nu}R^{\nu\mu\lambda}{}_{\kappa}\right) \\
&- \frac{1}{16}\frac{\delta}{\delta g}\int \left(R_{\mu\nu}{}^{\kappa}{}_{\lambda}R^{\lambda}{}_{\kappa}{}^{\mu\nu} - R_{\mu\nu}{}^{\kappa}{}_{\lambda}R_{\kappa}{}^{\lambda\mu\nu} - R_{\mu\nu}{}^{\kappa}{}_{\lambda}R^{\mu\nu\lambda}{}_{\kappa} + R_{\mu\nu}{}^{\kappa}{}_{\lambda}R^{\nu\mu\lambda}{}_{\kappa}\right) \\
&+ \frac{1}{16}\frac{\delta}{\delta g}\int \left(R_{\nu\mu}{}^{\kappa}{}_{\lambda}R^{\lambda}{}_{\kappa}{}^{\mu\nu} - R_{\nu\mu}{}^{\kappa}{}_{\lambda}R_{\kappa}{}^{\lambda\mu\nu} - R_{\nu\mu}{}^{\kappa}{}_{\lambda}R^{\mu\nu\lambda}{}_{\kappa} + R_{\nu\mu}{}^{\kappa}{}_{\lambda}R^{\nu\mu\lambda}{}_{\kappa}\right).
\end{aligned}
$$

Using Proposition A.1.1 and calculating these 16 variations separately, lengthy but straightforward calculations gives

$$
\begin{aligned}
&\frac{\delta}{\delta g}\int (R^{(5)}, R^{(5)})_{YM} \\
&= \int (\delta g_{\alpha\beta})\left(R_{\kappa\lambda\mu}{}^{\alpha}\left(R^{\mu\beta\lambda\kappa} - R^{\lambda\kappa\mu\beta}\right) + \frac{1}{4}R_{\lambda\kappa\mu\nu}\left(R^{\kappa\lambda\mu\nu} - R^{\mu\nu\kappa\lambda}\right)g^{\alpha\beta}\right).
\end{aligned}
$$

Using the explicit formula for curvature (2.54) and the results from Remark 1.4.10 and Lemma 1.4.11, we get that

$$\frac{\delta}{\delta g}\int (R^{(5)}, R^{(5)})_{YM}$$
$$= \int (\delta g_{\alpha\beta}) \left(\epsilon^{\xi\beta\lambda\alpha} Ric_{\lambda\mu} Ric_*{}^{\mu}{}_{\xi} + \epsilon^{\kappa\mu\xi\beta} W_{\kappa\mu\lambda}{}^{\alpha} Ric_*{}^{\lambda}{}_{\xi}\right). \tag{2.56}$$

Since $R^{(3)} = R - R^{(1)} - R^{(5)}$ we get that

$$\frac{\delta}{\delta g}\int (R^{(3)}, R^{(3)})_{YM}$$
$$= \frac{\delta}{\delta g}\int (R, R)_{YM} - \frac{\delta}{\delta g}\int (R^{(5)}, R^{(5)})_{YM} - \frac{\delta}{\delta g}\int (R^{(1)}, R^{(1)})_{YM}.$$

The variation $\frac{\delta}{\delta g}\int (R, R)_{YM}$ is given in Appendix A. Combining the formulae (2.54), (2.55), (2.56), (A.4) and using the results from Remark 1.4.10 and Lemma 1.4.11, we get that

$$\frac{\delta}{\delta g}\int (R^{(3)}, R^{(3)})_{YM}$$
$$= \int (\delta g_{\alpha\beta})(-2W^{\kappa\beta\alpha\nu} Ric_{\kappa\nu} + \epsilon^{\eta\kappa\alpha\nu} Ric_*{}^{\beta}{}_{\eta} Ric_{\kappa\nu} + \epsilon^{\kappa\mu\eta\alpha} W_{\kappa\mu\lambda}{}^{\beta} Ric_*{}^{\lambda}{}_{\eta}). \tag{2.57}$$

Combining the formulae (2.52), (2.55), (2.56), (2.57) and the Bianchi identity for the ${}^*\mathcal{W}$ we get the explicit representation (2.49) of the field equation (1.2).

2.3.3 Variation with respect to connection

The variations $\frac{\delta}{\delta\Gamma}\int (R^{(j)}, R^{(j)})_{YM}$ were calculated already in [99] and it was shown that

$$\frac{\delta}{\delta\Gamma}\int (R^{(j)}, R^{(j)})_{YM} = 4\int \left((\delta_{YM} R^{(j)})^{\mu} (\delta\Gamma)_{\mu}\right), \tag{2.58}$$

where

$$(\delta_{\rm YM} R)^{\mu} := \frac{1}{\sqrt{|\mathrm{det} g|}}\,(\partial_{\nu} + [\Gamma_{\nu}\,,\,\cdot\,])\left(\sqrt{|\det g|}\, R^{\mu\nu}\right)$$

is the Yang-Mills divergence. According to (2.58) and using the identity

$$\{\Gamma\}^{\xi}{}_{\xi\nu} = \frac{\partial_{\nu}|\det g|}{2|\det g|}, \tag{2.59}$$

we get that

$$\frac{\delta}{\delta\Gamma}\int (R^{(1)},R^{(1)})_{YM} = 4\int (\delta_{YM}R^{(1)})^{\mu}(\delta\Gamma)_{\mu} =$$
$$= 4\int (\delta\Gamma^{\kappa}{}_{\mu\lambda})\left(\partial_\nu {R^{(1)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \{\Gamma\}^{\xi}{}_{\xi\nu}{R^{(1)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \Gamma^{\lambda}{}_{\nu\eta}{R^{(1)}}^{\eta}{}_{\kappa}{}^{\mu\nu} - \Gamma^{\eta}{}_{\nu\kappa}{R^{(1)}}^{\lambda}{}_{\eta}{}^{\mu\nu}\right).$$

Using the definition of covariant derivative (1.10), identity (1.14) and the fact that torsion is purely axial, we get that

$$\frac{\delta}{\delta\Gamma}\int (R^{(1)},R^{(1)})_{YM} = 4\int (\delta\Gamma^{\lambda\mu\kappa})\left(\nabla_\nu R^{(1)}{}_{\kappa\lambda\mu}{}^{\nu} - \Gamma_{\mu\nu\eta}R^{(1)}{}_{\kappa\lambda}{}^{\eta\nu}\right).$$

Using metric compatibility, formula (1.31) and the fact that torsion is purely axial, we get that

$$\begin{aligned}\frac{\delta}{\delta\Gamma}\int (R^{(1)},R^{(1)})_{YM} = 2\int (\delta\Gamma^{\lambda\mu\kappa})(&\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + g_{\kappa\mu}\nabla_\xi Ric_{\lambda}{}^{\xi} \\ &- g_{\mu\lambda}\nabla_\xi Ric_{\kappa}{}^{\xi} + 2Ric_{\kappa}{}^{\eta}K_{\mu\eta\lambda} + 2Ric_{\lambda}{}^{\eta}K_{\mu\kappa\eta}).\end{aligned} \tag{2.60}$$

Using (2.58), (2.59) and the fact that torsion is purely axial, we get that

$$\begin{aligned}\frac{\delta}{\delta\Gamma}\int (R^{(3)},R^{(3)})_{YM} &= 4\int (\delta_{YM}R^{(3)})^{\mu}(\delta\Gamma)_{\mu} \\ &= 4\int (\delta\Gamma^{\kappa}{}_{\mu\lambda})\left(\partial_\nu {R^{(3)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \{\Gamma\}^{\xi}{}_{\xi\nu}{R^{(3)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \Gamma^{\lambda}{}_{\nu\eta}{R^{(3)}}^{\eta}{}_{\kappa}{}^{\mu\nu} - \Gamma^{\eta}{}_{\nu\kappa}{R^{(3)}}^{\lambda}{}_{\eta}{}^{\mu\nu}\right) \\ &= 4\int (\delta\Gamma^{\lambda\mu\kappa})\left(\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^{\nu} - K_{\mu\nu\eta}\mathcal{W}_{\kappa\lambda}{}^{\eta\nu}\right).\end{aligned} \tag{2.61}$$

Using (2.58), (2.59) and the fact that torsion is purely axial, we get that

$$\begin{aligned}\frac{\delta}{\delta\Gamma}\int (R^{(5)},R^{(5)})_{YM} &= 4\int (\delta_{YM}R^{(5)})^{\mu}(\delta\Gamma)_{\mu} = \\ &= 4\int (\delta\Gamma^{\kappa}{}_{\mu\lambda})\left(\partial_\nu {R^{(5)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \{\Gamma\}^{\xi}{}_{\xi\nu}{R^{(5)}}^{\lambda}{}_{\kappa}{}^{\mu\nu} + \Gamma^{\lambda}{}_{\nu\eta}{R^{(5)}}^{\eta}{}_{\kappa}{}^{\mu\nu} - \Gamma^{\eta}{}_{\nu\kappa}{R^{(5)}}^{\lambda}{}_{\eta}{}^{\mu\nu}\right) \\ &= 4\int (\delta\Gamma^{\lambda\mu\kappa})\left(\nabla_\nu R^{(5)}{}_{\kappa\lambda\mu}{}^{\nu} - \Gamma_{\mu\nu\eta}R^{(5)}{}_{\kappa\lambda}{}^{\eta\nu}\right).\end{aligned}$$

Using the explicit formula (B.3) for the $R^{(5)}$ piece of curvature and Lemma 1.4.9, we get that

$$\begin{aligned}\frac{\delta}{\delta\Gamma}\int (R^{(5)},R^{(5)})_{YM} = 2\int (\delta\Gamma^{\lambda\mu\kappa})\,(&-\epsilon^{\eta}{}_{\lambda\mu}{}^{\nu}\nabla_\nu Ric_{*\kappa\eta} + \epsilon^{\eta}{}_{\kappa\mu}{}^{\nu}\nabla_\nu Ric_{*\lambda\eta} \\ &+\epsilon^{\eta}{}_{\lambda}{}^{\xi\nu}K_{\mu\nu\xi}Ric_{*\kappa\eta} - \epsilon^{\eta}{}_{\kappa}{}^{\xi\nu}K_{\mu\nu\xi}Ric_{*\lambda\eta}).\end{aligned} \tag{2.62}$$

Combining the formulae (2.52), (2.60), (2.61) and (2.62) we get that the second field equation (1.3) is given explicitly by

$$\begin{aligned}
0 = & \, c_1(\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} + g_{\kappa\mu}\nabla_\xi Ric_\lambda{}^\xi - g_{\mu\lambda}\nabla_\xi Ric_\kappa{}^\xi) \\
& + 2c_1(K_{\mu\eta\lambda} Ric_\kappa{}^\eta + K_{\mu\kappa\eta} Ric_\lambda{}^\eta) + 2c_3\left(\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^\nu - K_{\mu\nu\eta}\mathcal{W}_{\kappa\lambda}{}^{\eta\nu}\right) \\
& + c_5\left(-\epsilon^\eta{}_{\lambda\mu}{}^\nu\nabla_\nu Ric_{*\kappa\eta} + \epsilon^\eta{}_{\kappa\mu}{}^\nu\nabla_\nu Ric_{*\lambda\eta}\right) \\
& + c_5\left(+\epsilon^\eta{}_\lambda{}^{\xi\nu}K_{\mu\nu\xi}Ric_{*\kappa\eta} - \epsilon^\eta{}_\kappa{}^{\xi\nu}K_{\mu\nu\xi}Ric_{*\lambda\eta}\right),
\end{aligned} \tag{2.63}$$

where c_1, c_3, c_5 are the coefficients of the quadratic form (1.4).

Now, using the Bianchi identity (B.5), we can explicitly express the terms ∇Ric and $\nabla\mathcal{W}$ containing one contraction, see Appendix B for detailed calculations, to get that

$$\nabla_\xi Ric^\xi{}_\lambda = - K^\eta{}_{\xi\zeta}\mathcal{W}^{\xi\zeta}{}_{\lambda\eta} - \epsilon^{\vartheta\zeta}{}_{\eta\lambda}K^\eta{}_{\xi\zeta}Ric_*{}^\xi{}_\vartheta \tag{2.64}$$

and

$$\begin{aligned}
\nabla_\mu \mathcal{W}^\mu{}_{\lambda\nu\xi} = & -\frac{1}{2}\Big(\nabla_\xi Ric_{\lambda\nu} - \nabla_\nu Ric_{\lambda\xi} + 2Ric^\mu{}_\xi K_{\nu\mu\lambda} + 2Ric^\mu{}_\nu K_{\mu\xi\lambda} \\
& + \epsilon^{\vartheta\mu}{}_{\nu\xi}\nabla_\mu Ric_{*\lambda\vartheta} - \epsilon^\vartheta{}_{\lambda\nu\xi}\nabla_\mu Ric_*{}^\mu{}_\vartheta\Big) \\
& - K^\eta{}_{\mu\xi}(\epsilon^\vartheta{}_{\lambda\eta\nu}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_{*\lambda\vartheta}) - K^\eta{}_{\mu\nu}(\epsilon^\vartheta{}_{\lambda\xi\eta}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\xi\eta}Ric_{*\lambda\vartheta}) \\
& + \frac{1}{2}K^\eta{}_{\mu\zeta}\left(g_{\lambda\xi}\epsilon^{\vartheta\zeta}{}_{\eta\nu}Ric_*{}^\mu{}_\vartheta - g_{\lambda\nu}\epsilon^{\vartheta\zeta}{}_{\eta\xi}Ric_*{}^\mu{}_\vartheta\right) \\
& + \frac{1}{2}K^\eta{}_{\mu\zeta}(g_{\lambda\xi}\mathcal{W}^{\mu\zeta}{}_{\nu\eta} - g_{\lambda\nu}\mathcal{W}^{\mu\zeta}{}_{\xi\eta}) - 2K^\eta{}_{\nu\mu}\mathcal{W}^\mu{}_{\lambda\xi\eta} - 2K^\eta{}_{\mu\xi}\mathcal{W}^\mu{}_{\lambda\nu\eta}.
\end{aligned} \tag{2.65}$$

Substituting (2.64) and (2.65) into (2.63), we exclude the terms ∇Ric and $\nabla\mathcal{W}$ from equation (2.63) and hence we get the explicit representation (2.50) of the field equation (1.3).

This concludes the proof of Theorem 2.3.10.

2.3.4 PP-type solutions of the field equations

The aim of this section is to prove Theorem 2.3.4, i.e. that *generalised pp-waves with purely axial torsion of parallel Ricci curvature are solutions of* (1.2), (1.3) *in the case* (1.4).

Proof *of Theorem 2.3.4.* We will prove this theorem by the direct substitution of the explicit formulae for torsion (2.34), (2.39), Weyl curvature (2.13) and the Ricci curvature (2.44) and Ric_* curvature (2.45) in the explicit representation of the field equations (2.49), (2.50). Also, we will use equations (2.8), (2.9) in order to simplify our calculations.

The terms $\mathcal{W} \times Ric$, $\mathcal{W} \times Ric_*$ and $Ric \times Ric_*$ with one contraction are zero since the vectors l, m_1 and m_2 are orthogonal, see (2.9). Hence, equation (2.49) is satisfied.

Let us now consider the equation (2.50). We will first consider the terms $T \times \mathcal{W}$ with two contractions. Using formulae (2.9), (2.13) and (2.39), we have that

$$\begin{aligned} T^\eta{}_{\lambda\xi}\mathcal{W}^\xi{}_{\mu\kappa\eta} &= \mp k(x^3)(l \wedge m_1 \wedge m_2)^\eta{}_{\lambda\xi} \sum_{j,k=1}^{2} w_{jk}(l \wedge m_j)^\xi{}_\mu (l \wedge m_k)_{\kappa\eta} \\ &= \mp k(x^3)(-w_{12} l_\lambda l_\mu l_\kappa + w_{21} l_\lambda l_\mu l_\kappa) = 0, \end{aligned}$$

since $w_{12} = w_{21}$. Consequently, the terms $T \times \mathcal{W}$ with three contractions are also zero.

The terms $T \times Ric$ with one contraction are equal to zero since

$$\begin{aligned} T_{\mu\eta\lambda} Ric_\kappa{}^\eta &= \frac{1}{2}\left(f_{11} + f_{22} - (k(x^3))^2\right) k(x^3)\, l^\xi\, \epsilon_{\xi\mu\eta\lambda}\, l_\kappa l^\eta \\ &= \frac{1}{2}\left(f_{11} + f_{22} - (k(x^3))^2\right) k(x^3)\, l^\xi l^\eta\, \epsilon_{\xi\eta\mu\lambda}\, l_\kappa = 0 \end{aligned}$$

as the product of the symmetric tensor $l^\xi l^\eta$ and the antisymmetric Levi-Civita tensor.

Similarly, the terms $T \times Ric_*$ with one contraction are equal to zero, since we get the product of a symmetric tensor and an antisymmetric tensor.

Now consider the terms $\epsilon \times T \times Ric_*$ with two contractions between ϵ and T and one contraction between ϵ and Ric_*. Using formulae (1.49), (2.34) and (2.45), we get that

$$\epsilon^{\vartheta\xi}{}_{\eta\lambda} T^\eta{}_{\xi\kappa} Ric_{*\mu\vartheta} = 2k(x^3)k'(x^3) l_\kappa l_\lambda l_\mu.$$

Hence

$$\epsilon^{\vartheta\xi}{}_{\eta\lambda} T^\eta{}_{\xi\kappa} Ric_{*\mu\vartheta} - \epsilon^{\vartheta\xi}{}_{\eta\kappa} T^\eta{}_{\xi\lambda} Ric_{*\mu\vartheta} = 0.$$

Now consider the terms $\epsilon \times \nabla Ric_*$. Since $\Gamma^\eta{}_{\xi\mu} Ric_{*\eta\vartheta} = 0$, we have that

$$\nabla_\xi Ric_{*\mu\vartheta} = \partial_\xi Ric_{*\mu\vartheta} - \Gamma^\eta{}_{\xi\mu} Ric_{*\eta\vartheta} - \Gamma^\eta{}_{\xi\vartheta} Ric_{*\mu\eta} = -k''(x^3) l_\xi l_\mu l_\vartheta.$$

Clearly $\nabla_\xi Ric_*{}^\xi{}_\eta = -k''(x^3) l_\xi l^\xi l_\eta = 0$. The only nonzero term of $\nabla_\xi Ric_{*\mu\vartheta}$ is $\nabla_3 Ric_{*33} = -k''(x^3)$ and consequently we have that the term $\epsilon \times \nabla Ric_*$ with two contractions between ϵ and ∇Ric_* are zero. This is because

$$\epsilon^{\vartheta\xi}{}_{\lambda\kappa} \nabla_\xi Ric_{*\mu\vartheta} = \epsilon^{33}{}_{\lambda\kappa} \nabla_3 Ric_{*33} = 0.$$

Equation (2.50) now reduces to checking that

$$\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} = 0. \tag{2.66}$$

Assuming that Ricci curvature is parallel, equation (2.66) is clearly satisfied. However, we see that parallel Ricci curvature is in fact not required, only the Cotton tensor of generalised pp-waves with purely axial torsion needs to vanish.

This ends the proof of Theorem 2.3.4. □

Remark 2.3.13. The Cotton tensor $\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu}$ of generalised pp-waves with purely axial torsion vanishes if and only if $f_{11} + f_{22} = \text{const}$. This condition is equivalent to the condition $\{\nabla\}\{Ric\} = 0$, i.e. that the Ricci curvature of the classical pp-wave is parallel.

2.4 The massless Dirac operator in theories of gravity

In this section we give the mathematical and physical significance of the spacetimes which are considered in Section 2.2.2 and we attempt to give a physical interpretation of these spacetimes as was done similarly in [74]. As we stated in [74], classical pp-waves of parallel Ricci curvature do not have an obvious physical interpretation and therefore they should not be viewed separately. Our analysis of the generalised pp-waves of parallel Ricci curvature shows that the classical pp-waves of parallel Ricci curvature are just the part of much wider class of solutions. Analysing the formula for curvature of the generalised pp-waves with purely axial torsion, we notice that the curvature in our special local coordinates (2.1), (2.5) is the sum of the curvature of the underlying classical pp-space

$$-\frac{1}{2}(l \wedge \{\nabla\}) \otimes (l \wedge \{\nabla\}) f \tag{2.67}$$

and the curvature

$$\frac{1}{4}(k(\varphi))^2 \operatorname{Re}\left((l \wedge m) \otimes (l \wedge \bar{m})\right) \mp \frac{1}{2} k'(\varphi) \operatorname{Im}\left((l \wedge m) \otimes (l \wedge \bar{m})\right) \tag{2.68}$$

generated by an axial torsion wave traveling over the pp-space. This remarkable property for curvature is not a trivial fact. Similarly, the property that curvatures just add up was also present in the case of generalised pp-waves with purely tensor torsion, see [74]. The physical interpretation of generalised pp-waves with purely tensor torsion introduced in Section 2.2.1 was given in [74] where it was proposed that generalised pp-waves with purely tensor torsion of parallel Ricci curvature represent a metric-affine model for the massless neutrino. Hence, in order to give physical interpretation of generalised pp-waves with purely axial torsion, we now compare these spacetimes to the solutions of Einstein-Weyl theory. Einstein-Weyl theory is the classical model which describes the interaction of gravitational and massless neutrino fields.

Remark 2.4.1. Our torsion and torsion generated curvature can be interpreted as waves traveling at speed of light. The underlying classical pp-space of parallel Ricci curvature can then be viewed as the gravitational imprint created by a wave of some massless matter field. As pointed out in [74], such a situation occurs in Einstein-Weyl theory.

In line with the traditions of quantum mechanics, we choose to complexify the curvature (2.68). The complexified curvature can be written as

$$\mathfrak{R}_A := r\,(l \wedge m) \otimes (l \wedge \overline{m}), \tag{2.69}$$

where

$$r := \frac{1}{4}(k(\varphi))^2 \pm \frac{\mathrm{i}}{2} k'(\varphi) \tag{2.70}$$

is a complex function. The function r is a function of the phase (2.4) and the curvature (2.68) is the real part of the complexified curvature (2.69). The curvature $\mathfrak{R}_A$ is polarized, i.e.

$$^*\mathfrak{R}_A = -\mathfrak{R}_A^* = \pm\mathrm{i}\,\mathfrak{R}_A,$$

where the $\pm$ sign depends on the sign of (2.5). It can also be written as

$$\mathfrak{R}_{A\,\alpha\beta\gamma\delta} = \sigma_{\alpha\beta ab}\,\omega^{abcd}\,\overline{\sigma}_{\gamma\delta cd} \tag{2.71}$$

where ω is some symmetric rank 4 spinor and $\sigma_{\alpha\beta}$ are second order Pauli matrices (2.16) where $\overline{\sigma}$ denotes their complex conjugation. The complex conjugate matrices $\overline{\sigma}$ are exactly the same set of matrices (2.16) only with the opposite sign chosen to correspond with the sign in (2.5).

Resolving (2.71) with respect to ω yields

$$\omega = \xi \otimes \xi \otimes \xi \otimes \xi, \tag{2.72}$$

where

$$\xi^a := r^{1/4}\,\chi^a. \tag{2.73}$$

Note that the spinor χ^a which appears in formula (2.73) is the parallel spinor field of the underlying pp-space (2.5) and the complex function r is given by (2.70).

Formula (2.72) shows that the rank 4 spinor ω is the 4th tensor power of a rank 1 spinor ξ. Hence, the curvature $\mathfrak{R}_A$ is completely determined by the rank 1 spinor field ξ.

Interestingly, we can now establish a connection between the generalised pp-waves with purely axial torsion and a massless neutrino field. A massless neutrino field is a metric compatible spacetime (with or without torsion) described by the action

$$S_{\text{neutrino}} := 2\mathrm{i}\int\Big(\xi^a\,\sigma^\mu{}_{a\dot b}\,(\nabla_\mu\bar\xi^{\dot b})\;-\;(\nabla_\mu\xi^a)\,\sigma^\mu{}_{a\dot b}\,\bar\xi^{\dot b}\Big), \tag{2.74}$$

see formula (11) of [44]. Varying the action (2.74) with respect to the spinor ξ, while keeping torsion and the metric fixed, we get the *massless Dirac* equation

$$\sigma^\mu{}_{a\dot b}\nabla_\mu\xi^a-\frac{1}{2}T^\eta{}_{\eta\mu}\sigma^\mu{}_{a\dot b}\xi^a=0, \tag{2.75}$$

which can equivalently be written as

$$\sigma^\mu{}_{a\dot b}\{\nabla\}_\mu\xi^a\pm\frac{i}{4}\varepsilon_{\alpha\beta\gamma\delta}T^{\alpha\beta\gamma}\sigma^\delta{}_{a\dot b}\xi^a=0, \tag{2.76}$$

where $\{\nabla\}$ is the covariant derivative with respect to the Levi-Civita connection, see Appendix B of [74]. Also, using the massless Dirac operator, see Section 3.2, the massless Dirac equation can be obtained by varying the action (3.12) with respect to the spinor ξ. The massless Dirac operator describes a single massless neutrino living in a compact universe and physically its eigenvalues are interpreted to be the energy levels of that massless particle. An interesting property is stated by the following

Lemma 2.4.2. *The spinor field (2.73) satisfies the massless Dirac equation.*

Proof. Since the classical pp-wave spacetime admits the parallel spinor field χ then

$$\sigma^\mu{}_{a\dot b}\{\nabla\}_\mu\xi^a=(r^{1/4})'\sigma^\mu{}_{a\dot b}l_\mu\chi^a=0$$

for the Pauli matrices (2.15) and the special local coordinates (2.5). Also, according to formulae (1.48), (2.34) and the Pauli matrices (2.15), we have that

$$\varepsilon_{\alpha\beta\gamma\delta}T^{\alpha\beta\gamma}\sigma^\delta{}_{a\dot b}\xi^a=\varepsilon_{\alpha\beta\gamma\delta}l_\mu k(\varphi)\varepsilon^{\mu\alpha\beta\gamma}\sigma^\delta{}_{a\dot b}\xi^a=6k(\varphi)l_\mu\sigma^\mu{}_{a\dot b}\xi^a=0,$$

i.e. the massless Dirac equation (2.76) is satisfied. □

In an attempt to give a physical interpretation of our generalised pp-waves with purely axial torsion, we provide the pp-wave type solution of Einstein-Weyl theory and compare them to the pp-wave type solutions of our conformally invariant metric-affine model of gravity.

2.4.1 Metric-affine vs Einstein-Weyl pp-wave type solutions

In Einstein-Weyl theory we consider the action as

$$S_{EW} := 2\mathrm{i}\int \left(\xi^a\, \sigma^\mu{}_{a\dot b}\, (\{\nabla\}_\mu \bar\xi^{\dot b}) \;-\; (\{\nabla\}_\mu \xi^a)\, \sigma^\mu{}_{a\dot b}\, \bar\xi^{\dot b} \right) + K \int \mathcal{R}, \tag{2.77}$$

where the constant $K = \frac{c^4}{16\pi G}$ is the universal constant where c is the speed of light and G is the gravitational constant, see [63]. In Einstein-Weyl theory the connection is assumed to be Levi-Civita, so we obtain the Einstein-Weyl field equations by varying the action (2.77) with respect to the metric and the spinor, i.e.

$$\frac{\delta S_{EW}}{\delta g} = 0, \tag{2.78}$$

$$\frac{\delta S_{EW}}{\delta \xi} = 0. \tag{2.79}$$

The variation of the first term of the action (2.77) with respect to the metric yields the energy momentum tensor of the Weyl action (2.74). For the detailed derivation of formula for the energy momentum tensor see Appendix B of [74]. The explicit representation of the Einstein-Weyl field equations (2.78), (2.79) is given by

$$\frac{\mathrm{i}}{2}\left[\sigma^\nu{}_{a\dot b} \left(\bar\xi^{\dot b} \{\nabla\}^\mu \xi^a - \xi^a \{\nabla\}^\mu \bar\xi^{\dot b} \right) + \sigma^\mu{}_{a\dot b} \left(\bar\xi^{\dot b} \{\nabla\}^\nu \xi^a - \xi^a \{\nabla\}^\nu \bar\xi^{\dot b} \right) \right]$$
$$+\mathrm{i} \left(\xi^a\, \sigma^\eta{}_{a\dot b}\, (\{\nabla\}_\eta \bar\xi^{\dot b}) g^{\mu\nu} - (\{\nabla\}_\eta \xi^a)\, \sigma^\eta{}_{a\dot b}\, \bar\xi^{\dot b} g^{\mu\nu} \right)$$
$$-K Ric^{\mu\nu} + \frac{K}{2} \mathcal{R} g^{\mu\nu} = 0, \tag{2.80}$$

$$\sigma^\mu{}_{a\dot b} \{\nabla\}_\mu\, \xi^a = 0. \tag{2.81}$$

The examination of the Einstein-Weyl field equations has a long history, see e.g. [7, 21, 22, 42, 43, 45, 46, 56]. A review of known solutions of Einstein-Weyl theory is given in [72, 74]. The nonlinear system of equations (2.80), (2.81) has solutions in the form of pp-waves, as stated in [72, 74].

We now wish to present a class of explicit solutions of (2.80), (2.81) where the metric g is in the form of a pp-metric (2.1) and the spinor ξ as in (2.73). The spinor (2.73) satisfies the massless Dirac equation (2.81), see Lemma 2.4.2. Because of this and the fact that the scalar curvature is zero in the setting of a pp-space, the equation (2.80) now becomes

$$\frac{\mathrm{i}}{2}\sigma^{\nu}{}_{a\dot{b}}\left(\bar{\xi}^{\dot{b}}\{\nabla\}^{\mu}\xi^{a}-\xi^{a}\{\nabla\}^{\mu}\bar{\xi}^{\dot{b}}\right)+\frac{\mathrm{i}}{2}\sigma^{\mu}{}_{a\dot{b}}\left(\bar{\xi}^{\dot{b}}\{\nabla\}^{\nu}\xi^{a}-\xi^{a}\{\nabla\}^{\nu}\bar{\xi}^{\dot{b}}\right)-K\,Ric^{\mu\nu}=0.$$

Substituting formulae (2.44), (2.73) into the above equation, we get that

$$\mathrm{i}(\sigma^{\nu}{}_{a\dot{b}}l^{\mu}+\sigma^{\mu}{}_{a\dot{b}}l^{\nu})\Big((r^{1/4})'\ \overline{r^{1/4}}-r^{1/4}\ (\overline{r^{1/4}})'\Big)\,\chi^{a}\bar{\chi}^{\dot{b}}=Kl^{\mu}l^{\nu}\big(f_{11}+f_{22}-k(x^{3})^{2}\big)\,.$$

The condition for a pp-wave type solution needs to satisfy in order to be a solution of Einstein-Weyl is

$$f_{11}+f_{22}=k(\varphi)^{2}+\frac{2\mathrm{i}}{K}\left((r^{1/4})'\ \overline{r^{1/4}}-r^{1/4}\ (\overline{r^{1/4}})'\right), \tag{2.82}$$

since $\sigma^{\mu}{}_{a\dot{b}}\chi^{a}\bar{\chi}^{\dot{b}}=l^{\mu}$. Since the function $k(\varphi)$ is arbitrary real function hence the complex function $r(\varphi)$ can be chosen arbitrarily and it uniquely determines the RHS of (2.82).

Remark 2.4.3. As was stated in [74], the main difference between the two models is that in the metric-affine model the generalised pp-wave solutions have parallel *Ric* curvature, whereas in the Einstein-Weyl model the pp-wave type solutions do not necessarily have parallel Ricci curvature.

The comparison of this two types of solutions becomes much clearer in the case of the monochromatic solutions. Similarly as was done in [74], if we choose the function $k(x^3)$ such that the function (2.70) is equal to

$$r=c^{4}e^{4\mathrm{i}(ax^{3}+b)},$$

where $a,b,c\in\mathbb{R},\ a\neq 0$, then the the spinor ξ form (2.73) is explicitly given by

$$\xi=c\begin{pmatrix}1\\0\end{pmatrix}e^{\mathrm{i}(ax^{3}+b)}. \tag{2.83}$$

The vector field A from Definition 2.2.5 is $A=k(x^3)l$ where the function $k(x^3)$ is the solution of the differential equation

$$\frac{1}{4}(k(x^{3}))^{2}\pm\frac{\mathrm{i}}{2}k'(x^{3})=c^{4}e^{4\mathrm{i}(ax^{3}+b)}$$

where $a, b, c \in \mathbb{R},\ a \neq 0$. For the spinor field (2.83), the condition (2.82) reads

$$f_{11} + f_{22} = (k(x^3))^2 - \frac{4ac^2}{K}.$$

Hence, we conclude that in the metric-affine case the Laplacian of f can be any constant, while in the Einstein-Weyl case it is required for it to be a particular constant, which is the consequence of conformal invariance of the metric-affine model and the presence of the gravitational constant in the Einstein-Weyl.

The generalised pp-waves of parallel Ricci curvature are very similar to pp-type solutions of the Einstein-Weyl model. According this conclusion, similarly to [74], we propose that *generalised pp-waves with purely axial torsion of parallel Ricci curvature represent a metric-affine model for the massless neutrino.*

Chapter 3

Spectral Analysis of the Massless Dirac Operator on a 3-dimensional Manifold

Generalised pp-waves with purely axial and purely tensor torsion of parallel Ricci curvature considered in this thesis have their particular physical interpretation as was shown in the previous chapter. The spinor field ξ, which completely determines the complexified curvatures of those spacetimes, satisfies the massless Dirac equation. The axial torsion considered in previous sections is the irreducible piece of torsion which is usually used when one models the massless neutrino, see [20], or the electron, see [16], by means of Cosserat elasticity. Now we are interested in a more mathematical approach and the analysis of the massless Dirac equation and the massless Dirac operator in 3 dimensions.

In this chapter we consider the massless Dirac operator on a 3-manifold which describes a single massless neutrino living in a 3-dimensional compact universe. The eigenvalues of the massless Dirac operator are interpreted to be the energy levels of that massless particle. The eigenvalues of the massless Dirac operator can be explicitly calculated when we consider the unit torus $\mathbb{T}^3$ equipped with Euclidean metric and the unit sphere $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$. It turns out that in these cases the spectrum of the massless Dirac operator is symmetric.

However, according to [3, 4, 5, 6], for a general oriented Riemannian 3-manifold (M, g) there is no physical reason for the spectrum of the massless Dirac operator to be symmetric. Physically interpreted, this symmetry would mean that in these two examples, the massless neutrino and the massless antineutrino have the same properties.

Our goal is to prove that it is possible break the spectral symmetry of the massless Dirac operator on a 3-torus using the perturbation of the Euclidean metric. We found justification for doing this in the numerical analysis of the spectrum of the massless Dirac operator. We use the Galerkin method, see e.g. [49], to calculate the spectrum explicitly for different perturbations of the Euclidean metric. Perturbing the Euclidean metric for some positive real parameter ϵ, we derive the asymptotic formulae for the eigenvalues of the massless Dirac operator, which depends to the parameter ϵ. We analyse under which perturbations of the metric it is possible to obtain the spectral asymmetry.

In the spectral analysis of the massless Dirac operator we will use some known results for an elliptic self-adjoint first-order differential operator, see [18, 19, 25].

3.1 Some properties of an elliptic self-adjoint first-order differential operator

One way to analyse the spectrum of an operator is to consider the distribution of the eigenvalues of that operator. Hence, we will be interested in studying the *spectral function*, see Definition 3.1.5, and *the counting function*, see Definition 3.1.6, of the massless Dirac operator. The massless Dirac operator is a self-adjoint first order differential operator and it has a discrete spectrum with eigenvalues accumulating to $+\infty$ and $-\infty$, while the eigenfunctions of the operator are infinitely smooth, see [19, 24, 25].

Let M be a connected compact 3-dimensional manifold without boundary and let $x = (x^1, x^2, x^3)$ be the local coordinates on the manifold M. Consider a first order differential operator A which is self-adjoint acting on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued half-densities over a manifold M.

The *principal* symbol and the *subprincipal* symbol of the first order differential operator A, which will be used through this chapter, are defined as follows, see [85].

Definition 3.1.1. The principal symbol of the first order differential operator A is a matrix obtained by leaving in A only the leading first order derivatives and replacing each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, 2, 3$, where $\xi = (\xi_1, \xi_2, \xi_3)$ is the variable dual to the position variable x which in physics literature is referred to as *momentum*. We denote the principal symbol of the operator A by $A_1(x, \xi)$.

As was shown in [19], the existence of a principal symbol $A_1(x, \xi)$ implies that our manifold M is parallelisable and the principal symbol admits the metric and the teleparallel connection $\Gamma^{\alpha}{}_{\beta\gamma}(x)$. This allows us to express the results of our spectral analysis in a geometric language and to define the torsion tensor by (1.12).

Definition 3.1.2. The subprincipal symbol of the first order differential operator A is defined as

$$A_{\mathrm{sub}} := A_0 + \frac{\mathrm{i}}{2}(A_1)_{x^\alpha \xi_\alpha}, \tag{3.1}$$

where $A_1(x,\xi)$ and $A_0(x)$ are the components of the full symbol $A(x,\xi) = A_1(x,\xi) + A_0(x)$ of our first order differential operator, with the subscript indicating the degree of homogeneity.

Remark 3.1.3. We assume that the principal symbol $A_1(x,\xi)$ is trace free for all $(x,\xi) \in T^*M$ and that $\det A_1(x,\xi) \neq 0$, $\forall\ (x,\xi) \in T'M$ where $T'M := T^*M\backslash\{\xi = 0\}$. The principal symbol of our operator A is a 2×2 Hermitian matrix-function on the cotangent bundle T^*M and linear in ξ.

Remark 3.1.4. It is known, see [18, 19], that under these assumptions, the spectrum of the operator A is discrete with eigenvalues accumulating to $\pm\infty$.

We denote by λ_k the eigenvalues of the operator A and by $v_k(x)$ the corresponding eigenvectors. The eigenvalues λ_k are enumerated in increasing order using $k = 1, 2, \ldots$ for positive eigenvalues and $k = 0, -1, -2, \ldots$ for non-positive eigenvalues.

For the purposes of further analysis of the spectrum of the operator, we are interested in the analysis of two functions, namely the spectral function and the counting function, which are defined as follows, see [19, 85].

Definition 3.1.5. The spectral function is the real density defined as

$$e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \| \, v_k(x) \, \|^2, \tag{3.2}$$

where $\| \, v_k(x) \, \|^2 := [v_k(x)]^* \, v_k(x)$ is the square of the Euclidean norm of the eigenfunction v_k evaluated at the point $x \in M$ and λ is a positive parameter (*spectral* parameter).

Definition 3.1.6. The counting function is the function

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 = \int_M e(\lambda, x, x)\,dx, \tag{3.3}$$

where $e(\lambda, x, x)$ is the spectral function (3.2).

The counting function $N(\lambda)$ is actually the number of eigenvalues between zero and λ. Also, we are interested in the asymptotic formulae of the spectral function (3.2) and the counting function (3.3), i.e. we are interested in deriving the formulae of the type

$$e(\lambda, x, x) = a(x)\lambda^3 + b(x)\lambda^2 + o(\lambda^2), \tag{3.4}$$

$$N(\lambda) = a\lambda^3 + b\lambda^2 + o(\lambda^2) \tag{3.5}$$

as $\lambda \to +\infty$, where the real constants $a,\ b$ and real densities $a(x),\ b(x)$ are related in accordance with

$$a = \int_M a(x)dx, \tag{3.6}$$

$$b = \int_M b(x)dx. \tag{3.7}$$

The asymptotic formulae (3.4) and (3.5) for the first order differential operator are explicitly derived in the following theorem.

Theorem 3.1.7. *The coefficients in the two-term asymptotic formula (3.4) are given by*

$$a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)},$$

$$b(x) = \frac{1}{8\pi^2}\left(\left(3c * T^{\mathrm{ax}} - 2\mathrm{tr}A_{\mathrm{sub}}\right)\sqrt{\det g_{\alpha\beta}}\right)(x),$$

where

$$T^{\mathrm{ax}}_{\alpha\beta\gamma} := \frac{1}{3}\left(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha}\right)$$

is axial torsion, c is topological charge defined by

$$c := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \mathrm{tr}((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}),$$

which can only take two values, $+1$ or -1 and where $*$ is the Hodge star operator (1.21).

For the proof of Theorem 3.1.7 see e.g. [19].

3.2 The massless Dirac operator

In this section we introduce the massless Dirac operator on a 3-dimensional manifold and we recall its main properties, as was similarly done in [19, 24].

For more on how the massless Dirac operator acts on a manifold of arbitrary dimension, see [36, 41].

Let M be a 3-dimensional connected compact oriented manifold equipped with a Riemannian metric $g_{\alpha\beta}$, where $\alpha, \beta = 1, 2, 3$ are tensor indices. According to [54], a 3-dimensional oriented manifold is parallelisable and consequently, there exist smooth real vector fields $e_j(x)$, $j = 1, 2, 3$ that are linearly independent in every point x of the manifold M.

The triple of linearly independent vector fields $e_j(x)$, $j = 1, 2, 3$, is called a *frame*. We can assume that the vector fields $e_j(x)$ are orthonormal, and if not, the orthonormality can always be achieved using the Gram-Schmidt process. The coordinate components of the vector $e_j(x)$ are $e_j{}^\alpha(x)$, $\alpha = 1, 2, 3$, where the so-called anholonomic or *frame index*, denoted by the Latin letter j, enumerates the vector field and the holonomic or *tensor index*, denoted by Greek letter α, enumerates their components.

The *coframe* is defined as the triple of covector fields $e^k(x)$, $k = 1, 2, 3$, and the coordinate components of the vector $e^k(x)$ are $e^k{}_\alpha(x)$, $\alpha = 1, 2, 3$, where

$$e^k{}_\beta := \delta^{kj} g_{\beta\gamma} e_j{}^\gamma.$$

The frame is completely determined by the coframe, and vice versa, by the relation $e_j{}^\alpha e^k{}_\alpha = \delta_j{}^k$.

Definition 3.2.1. The massless Dirac operator is defined to be the matrix operator

$$W := -\mathrm{i}\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) \right), \tag{3.8}$$

where summation is carried out over $\alpha, \beta, \gamma = 1, 2, 3$.

Remark 3.2.2. We denote the massless Dirac operator with the Latin letter "W" because in theoretical physics literature it is often referred to as the *Weyl operator*.

Remark 3.2.3. The massless Dirac operator can be thought of as a square root of the Laplacian.

The coefficients $\left\{ {\alpha \atop \beta\gamma} \right\}$ appearing in the Definition 3.2.1 are the Christoffel symbols (1.11). The matrices σ are the Pauli matrices defined as

$$\sigma^\alpha(x) := s^j e_j{}^\alpha(x), \tag{3.9}$$

where summation is carried out over the repeated frame index $j = 1, 2, 3$, the index $\alpha = 1, 2, 3$ is the free tensor index and the matrices s^j and s_j,

$j = 1, 2, 3$ are defined with

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1,\ s^2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = s_2,\ s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3. \quad (3.10)$$

The massless Dirac operator (3.8) acts on 2-columns

$$v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T \quad (3.11)$$

of complex-valued scalar functions which are referred to as the Weyl spinor. The spinor (3.11) transforms in a particular way under transformations of the orthonormal frame $e_j(x)$. We choose the frame *a priori* so that we can treat the components of the spinor as scalars.

The raising and of lowering indices follows the standard convention using the metric tensor.

Using the massless Dirac operator (3.8) we can construct *the massless Dirac action* - the variational functional corresponding to the operator (3.8) which we consider in Section 2.4.

Definition 3.2.4. The massless Dirac action is defined as

$$S(\zeta) := \int_M \mathrm{Re}(\zeta^* W \zeta) \sqrt{\det g_{\alpha\beta}}\, dx, \quad (3.12)$$

where W is the massless Dirac operator (3.8) and the star indicates Hermitian conjugation.

The orientation (positive or negative) of the massless Dirac operator (3.8) is completely determined by the sign of the frame. Our frame has positive orientation if $\det e_j > 0$ and negative orientation if $\det e_j < 0$. Note that the transformation $W \mapsto -W$ changes the orientation of the massless Dirac operator.

Physically interpreted, the operator (3.8) describes a single massless neutrino living in a 3-dimensional compact universe M and the energy levels of such particle are determined by the eigenvalues of the operator. We now list the main properties of the massless Dirac operator (3.8), see [17, 29] for their proofs:

- The massless Dirac operator is invariant under changes of local coordinates.
- The massless Dirac operator is an elliptic operator.

- The massless Dirac operator is formally self-adjoint with respect to the inner product
$$\langle v, w\rangle := \int_M w^* v\sqrt{\det g_{\alpha\beta}}dx \tag{3.13}$$
on 2-columns of smooth scalar functions $v, w : M \to \mathbb{C}^2$.

- It is of a special interest in theoretical physics to view the antilinear map called the *charge conjugation*, which is defined by
$$v \mapsto C(v) := \epsilon\overline{v}, \tag{3.14}$$
where
$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
The antilinear operator of charge conjugation (3.14) maps any element of $L^2(M;\mathbb{C}^2)$ to an element of $L^2(M;\mathbb{C}^2)$ and any element of $H^1(M;\mathbb{C}^2)$ to an element of $H^1(M;\mathbb{C}^2)$.

  Remark 3.2.5. An interesting property is that the massless Dirac operator W commutes with the charge conjugation operator, i.e.
$$C(Wv) = W(C(v)). \tag{3.15}$$

- Let $R : M \to SU(2)$ be an arbitrary smooth special unitary matrix-function. Let us introduce new Pauli matrices
$$\widetilde{\sigma}^\alpha := R\sigma^\alpha R^* \tag{3.16}$$
and a new operator $\widetilde{W}$ which is obtained by replacing the σ by $\widetilde{\sigma}$ in (3.8). Then, the operators W and $\widetilde{W}$ are related in the same way as the Pauli matrices σ and $\widetilde{\sigma}$, i.e.
$$\widetilde{W} := RWR^*.$$
If there exists a smooth matrix-function $R : M \to SU(2)$ such that the corresponding Pauli matrices σ^α and $\widetilde{\sigma}^\alpha$ are related in accordance with (3.16), we say that the operators W are $\widetilde{W}$ are *equivalent*.

The charge conjugation operator also possesses some additional interesting properties, as stated by following

Lemma 3.2.6. *The formulae* (3.13) *and* (3.14) *imply the following useful identities:*

$$C(C(v)) = -v, \tag{3.17}$$
$$\langle v, C(v)\rangle = 0, \tag{3.18}$$
$$\langle C(v), C(w)\rangle = \langle w, v\rangle. \tag{3.19}$$

Proof. Let $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ and $w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^T$ be arbitrary smooth scalar functions and

$$C(v) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{v_1} \\ \overline{v_2} \end{pmatrix} = \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}.$$

Then

$$C(C(v)) = C \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -v.$$

Also

$$\begin{aligned} \langle v, C(v)\rangle &= \int_M (C(v))^* v \sqrt{\det g_{\alpha\beta}}\; dx = \int_M \begin{pmatrix} -v_2 & v_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sqrt{\det g_{\alpha\beta}}\; dx \\ &= \int_M (-v_1 v_2 + v_1 v_2) \sqrt{\det g_{\alpha\beta}}\; dx = 0, \end{aligned}$$

and

$$\begin{aligned} \langle C(v), C(w)\rangle - \langle w, v\rangle &= \int_M (C(w))^* C(v) \sqrt{\det g_{\alpha\beta}}\; dx - \int_M (v)^* w \sqrt{\det g_{\alpha\beta}}\; dx \\ &= \int_M (w_2\overline{v_2} + w_1\overline{v_1} - \overline{v_1}w_1 - \overline{v_2}w_2) dx = 0. \end{aligned}$$

Hence, $\langle C(v), C(w)\rangle = \langle w, v\rangle$. ☐

As was stated in [19], the justification for the introduction of the half-density $\sqrt{\det g_{\alpha\beta}}$ in the formula (3.13) for the inner product lies in the fact that the massless Dirac operator (3.8) is an operator which acts as on 2-columns of scalar functions, i.e. on 2-columns of quantities which do not change under changes of local coordinates. The density and the half-density are defined as follows, see [85].

Definition 3.2.7. We say that μ is a *density* if $\mu(x) = J(x)\widetilde{\mu}(\widetilde{x}(x))$ and a *half-density* if $\mu(x) = J^{1/2}(x)\widetilde{\mu}(\widetilde{x}(x))$ where $\widetilde{\mu}$ is the representation of μ in coordinates $\widetilde{x}$ and $J(x) = |\det \partial\widetilde{x}/\partial x|$.

An operator of special interest of us is *the massless Dirac operator on half-densities*:

Definition 3.2.8. The massless Dirac operator on half-densities is the operator

$$W_{1/2} := (\det g_{\kappa\lambda})^{1/4} W (\det g_{\mu\nu})^{-1/4} \tag{3.20}$$

which maps half-densities to half-densities.

Remark 3.2.9. The operator (3.20) is equivalent to the massless Dirac operator (3.8).

The domain of the operator $W_{1/2}$ is $H^1(M; \mathbb{C}^2)$, which is the Sobolev space of 2-columns of half-densities that are square integrable together with their first partial derivatives.

3.3 The spectrum of the massless Dirac operator

We are now interested to apply the results from Section 3.1 for the first order differential operator to the spectral analysis of the massless Dirac operator (3.8).

We can now ask the question: which conditions need the first order differential operator A satisfy in order for it to be the massless Dirac operator on half-densities? The answer to that question is stated by following theorem.

Theorem 3.3.1. *The operator A is a massless Dirac operator on half-densities if and only if the following two conditions are satisfied at every point of the manifold M:*

- *the subprincipal symbol* (3.1) *of the operator is proportional to the identity matrix;*
- *the second asymptotic coefficient of the spectral function $b(x)$ is zero.*

For the proofs of Theorem 3.1.7 and Theorem 3.3.1, see [19].

The explicit formula for the principal symbol of the massless Dirac operator on half-densities (3.20) is given by

$$A_1(x,\xi) = \begin{pmatrix} {e_3}^\alpha & {e_1}^\alpha - i{e_2}^\alpha \\ {e_1}^\alpha + i{e_2}^\alpha & -{e_3}^\alpha \end{pmatrix} \xi_\alpha \tag{3.21}$$

and the explicit formula for the its subprincipal symbol is given by

$$A_{\mathrm{sub}}(x) = \frac{3}{4}(*T^{\mathrm{ax}}(x))I, \tag{3.22}$$

where

$$\begin{aligned} *T^{\mathrm{ax}}(x) = \frac{\delta_{lk}}{3}\sqrt{\det g^{\alpha\beta}}\,\big[&{e^k}_1\,\partial {e^l}_3/\partial x^2 + {e^k}_2\partial {e^l}_1/\partial x^3 + {e^k}_3\partial {e^l}_2/\partial x^1 \\ &-{e^k}_1\partial {e^l}_2/\partial x^3 - {e^k}_2\partial {e^l}_3/\partial x^1 - {e^k}_3\partial {e^l}_1/\partial x^2\big] \end{aligned} \tag{3.23}$$

is the explicit formula for the Hodge dual of the axial part of torsion, see Section 8 of [19].

The massless Dirac operator on half-densities (3.20) is a self-adjoint first order elliptic differential operator acting on 2-columns of complex-valued half-densities, $\det A_1(x,\xi) \neq 0$ and the principal symbol (3.21) is clearly trace free. Hence, the massless Dirac operator on half-densities satisfies the conditions for the differential operator A from Remark 3.1.3.

According to Theorem 3.1.7, the asymptotic formulae (3.4) and (3.5) for the massless Dirac operator on half-densities (3.20) read

$$e(\lambda,x,x) = \frac{\sqrt{\det g_{\alpha\beta}(x)}}{6\pi^2}\lambda^3 + o(\lambda^2), \tag{3.24}$$

$$N(\lambda) = \frac{\mathrm{Vol}\ M}{6\pi^2}\lambda^3 + o(\lambda^2), \tag{3.25}$$

where Vol M is the volume of the Riemannian 3-manifold M.

Remark 3.3.2. We want to stress the remarkable simplicity of the asymptotic formula (3.25) for the counting function of the operator (3.20). The asymptotic coefficient (3.6) is determined only by the volume of the Riemannian 3-manifold M and does not depend on the shape of the manifold M. The asymptotic coefficient (3.7) is equal to zero.

Remark 3.3.3. Analysing the formula (3.25) we can conclude that the positive eigenvalues of the massless Dirac operator are distributed in same way as the negative eigenvalues.

Remark 3.3.4. The factor $\sqrt{\det g_{\alpha\beta}(x)}$ appears in formula (3.24) because we are working with the massless Dirac operator on half-densities (3.20). Of course, for the massless Dirac operator on spinors (3.8) the spectral function is a scalar field and formula (3.24) reads

$$e(\lambda,x,x) = \frac{1}{6\pi^2}\lambda^3 + o(\lambda^2).$$

Example 3.3.5. Consider the unit torus $\mathbb{T}^3$ and unit sphere $\mathbb{S}^3$ as manifolds M. Since Vol $\mathbb{T}^3 = (2\pi)^3$ and Vol $\mathbb{S}^3 = 2\pi^2$ the counting functions (3.25) on a 3-torus and 3-sphere are respectively given by

$$N(\lambda) = \frac{4}{3}\pi\lambda^3 + o(\lambda^2), \tag{3.26}$$

and

$$N(\lambda) = \frac{1}{3}\lambda^3 + o(\lambda^2). \tag{3.27}$$

The counting functions (3.26) and (3.27) show that for these two choices of manifolds the positive eigenvalues are distributed in same way as the negative eigenvalues.

The relation between the spectrum and the geometry of the manifold is an object of intense research today. It is a very difficult task to determine the spectrum of the massless Dirac operator on an arbitrary manifold M. To our knowledge, the first explicit calculation of the spectrum of the Dirac operator was done by Friedrich [35]. In the same paper, the dependence of the spectrum to the choice of the spin structure was also shown.

For the moment, we can only analyse the counting function (3.25) to see the distribution of the eigenvalues. There are only two examples where the spectrum is determined explicitly. The first example is the unit torus $\mathbb{T}^3$ equipped with Euclidean metric and the second example is the unit sphere $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$. In both examples the spectrum is symmetric about zero, see [10, 19, 93].

To our knowledge, there are only two known examples where the spectrum of the massless Dirac operator can be calculated explicitly. The spectrum of the massless Dirac operator on the unit torus $\mathbb{T}^3$ equipped with Euclidean metric is as follows: zero is an eigenvalue of multiplicity two and for each $m \in \mathbb{Z}^3 \setminus \{0\}$ the eigenvalues are $\pm\|m\|$. The spectrum of the massless Dirac operator on $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$ can also be calculated explicitly, see [10, 93], and the eigenvalues are $\pm\left(k + \frac{1}{2}\right)$, $(k = 1, 2, \ldots)$, with multiplicity $k(k+1)$.

The eigenvalues of the massless Dirac operator have a very interesting property given by the following lemma, which is a consequence of the commuting of the massless Dirac operator and the charge conjugation operator.

Lemma 3.3.6. *The eigenvalues of the massless Dirac operator have even multiplicity.*

Proof. Let the vector v be the eigenvector corresponding to the eigenvalue λ of the massless Dirac operator, i.e. $Wv = \lambda v$. Let C be the charge conjugation operator. Then, according to the formula (3.15) we have that

$$W(C(v)) = C(Wv) = C(\lambda v) = \lambda C(v),$$

i.e. the vector $C(v)$ is also the eigenvector of the massless Dirac operator corresponding the same eigenvalue λ. $\square$

Remark 3.3.7. As was stated in [24], if we introduce the magnetic field as Erdős and Solovej did in [29], the situation is different and the double eigenvalues split up. This indicates that the double eigenvalues of the massless Dirac operator correspond to two different spins.

An interesting result for the lower eigenvalue estimate on closed manifolds is given by the so-called *Friedrich's inequality.*

Theorem 3.3.8. *If λ is an eigenvalue of the massless Dirac operator on an $n-$dimensional $(n \geqslant 2)$ closed Riemannian manifold (M^n, g) with a spin structure, then*

$$\lambda^2 \geqslant \frac{n}{4(n-1)} \inf_M(\mathcal{R}), \tag{3.28}$$

where $\mathcal{R}$ is scalar curvature.

For the proof of Theorem 3.3.8 and the improvements of the formula (3.28) see [36].

According to [3, 4, 5, 6], for a general oriented Riemannian 3-manifold (M, g) there is no reason for the spectrum of the massless Dirac operator to be symmetric. In this chapter we break the spectral symmetry working on the unit torus $\mathbb{T}^3$ as a 3-manifold with trivial topology perturbing the Euclidean metric. This approach was also used in [24].

3.4 The perturbation theory for the massless Dirac operator

One way to break the spectral symmetry of the massless Dirac operator is to use a trivial metric and a manifold with a nontrivial topology. In this chapter we use an opposite strategy: we use a manifold with a trivial topology and a nontrivial metric, i.e. we perturb the Euclidean metric in order to create the spectral asymmetry. In this section we develop the perturbation theory which is used for that purpose.

We consider perturbations of the Euclidean metric with a small positive real parameter ϵ. In order to have a simpler approach, we introduce a different approach to the frame and coframe because we are working in a specified coordinate system, in a similar fashion to [24]. For the Euclidean metric, it is easy to see that the massless Dirac operator (3.8) corresponding to the standard spin structure reads

$$W = -\mathrm{i}\begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix}. \tag{3.29}$$

Consider the perturbed metric $g_{\alpha\beta}(x;\epsilon)$ whose components are smooth functions of coordinates x^α and of small positive real parameter ϵ, which satisfies

$$g_{\alpha\beta}(x;0) = \delta_{\alpha\beta}. \tag{3.30}$$

Let

$$h_{\alpha\beta}(x) := \left.\frac{\partial g_{\alpha\beta}(x)}{\partial\epsilon}\right|_{\epsilon=0}, \qquad k_{\alpha\beta}(x) := 4\left.\frac{\partial^2 g_{\alpha\beta}(x)}{\partial\epsilon^2}\right|_{\epsilon=0}. \tag{3.31}$$

A perturbed coframe is a smooth real-valued matrix-function ${e^j}_\alpha(x;\epsilon)$, $j,\alpha = 1,2,3$ satisfying the conditions

$$g_{\alpha\beta}(x;\epsilon) = \delta_{jk}{e^j}_\alpha(x;\epsilon){e^k}_\beta(x;\epsilon), \tag{3.32}$$

$${e^j}_\alpha(x;0) = {\delta^j}_\alpha. \tag{3.33}$$

The reason for the condition (3.33) is that we want our unperturbed operator to have the form (3.29). The first index j of the matrix-function ${e^j}_\alpha(x;\epsilon)$ enumerates the matrix rows and the second index α enumerates the matrix columns. Since we are working in a specified coordinate system, we can require the coframe ${e^j}_\alpha(x;\epsilon)$ to be symmetric, i.e.

$${e^j}_\alpha(x;\epsilon) = {e^\alpha}_j(x;\epsilon). \tag{3.34}$$

The condition (3.34) will significantly simplify our calculations involving the massless Dirac operator. As was stated in [24], the asymptotic expansion of the subprincipal symbol of the massless Dirac operator on half-densities (3.22) in powers of ϵ starts with a quadratic term and that coefficient has a very simple structure. For the perturbed metric $g_{\alpha\beta}(x;\epsilon)$ the coframe ${e^j}_\alpha(x;\epsilon)$ is not determined uniquely. We can multiply the matrix-function ${e^j}_\alpha(x;\epsilon)$ from the left by an arbitrary smooth 3×3 special orthogonal matrix-function $O(x;\epsilon)$ satisfying the condition $O(x;0) = I$, with I denoting the 3×3 identity matrix. The new coframe will also satisfy the conditions (3.32), (3.33). This choice of the coframe does not affect the spectrum of the massless Dirac operator, see [19] for more details.

The frame is a smooth real-valued matrix-function $e_j{}^\alpha(x;\epsilon)$ defined by the system of linear algebraic equations

$$e_j{}^\alpha(x;\epsilon)e^k{}_\alpha(x;\epsilon) = \delta_j{}^k. \tag{3.35}$$

Remark 3.4.1. In matrix notation, formula (3.35) reads as *the frame is the transpose of the inverse of the coframe.*

As we chose our coframe to be symmetric, our frame is symmetric as well. According to formulae (3.30) and (3.31) we have that

$$g_{\alpha\beta}(x;\epsilon) = \delta_{\alpha\beta} + \epsilon h_{\alpha\beta}(x) + \frac{\epsilon^2}{4}k_{\alpha\beta}(x) + O(\epsilon^3). \tag{3.36}$$

Using Taylor's formula for the function $\sqrt{1+z}$, the coframe is then given by

$$e^j{}_\alpha(x;\epsilon) = \delta^j{}_\alpha + \frac{\epsilon}{2}h^j{}_\alpha(x) - \frac{\epsilon^2}{8}\left(h^2\right)^j{}_\alpha(x) + \frac{\epsilon^2}{8}k^j{}_\alpha(x) + O(\epsilon^3). \tag{3.37}$$

Using formula (3.37) and the Taylor's formula for $(1+z)^{-1}$, the frame is now given by

$$e_j{}^\alpha(x;\epsilon) = \delta_j{}^\alpha - \frac{\epsilon}{2}h_j{}^\alpha(x) + \frac{3\epsilon^2}{8}\left(h^2\right)_j{}^\alpha(x) - \frac{\epsilon^2}{8}k_j{}^\alpha(x) + O(\epsilon^3). \tag{3.38}$$

Remark 3.4.2. Note that h^2 appearing in formulae (3.37) and (3.38) denotes the square of the perturbation matrix h, i.e. $\left(h^2\right)_j{}^\alpha(x) = h_j{}^\beta(x)h_\beta{}^\alpha(x)$, where the summation is over the repeated index β.

We raise and lower indices in the matrices h and k using the Euclidean metric, which means that raising or lowering an index does not change anything.

We cannot apply standard perturbation theory for the massless Dirac operator, because the massless Dirac operator commutes with the charge conjugation operator and the eigenvalues have even multiplicity, see Lemma 3.3.6. In this section we present the perturbation theory which takes care of multiplicity of the eigenvalues. In the presence of a magnetic field, we can apply standard perturbation theory because the magnetic field splits up the double eigenvalues, see Remark 3.3.7.

Let $W_{1/2}(\epsilon)$ therefore be the massless Dirac operator on half-densities (3.20) corresponding to the perturbed metric $g_{\alpha\beta}(x;\epsilon)$. We prefer to deal with the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$ rather than the massless Dirac operator $W(\epsilon)$, although the spectra of these operators are the same, because $W_{1/2}(\epsilon)$ and $W(\epsilon)$ are equivalent.

Let

$$W_{1/2}(\epsilon) = W_{1/2}^{(0)} + \epsilon W_{1/2}^{(1)} + \epsilon^2 W_{1/2}^{(2)} + \cdots \tag{3.39}$$

be the asymptotic expansion of the perturbed massless Dirac operator in powers of the small parameter ϵ. The operator $W_{1/2}^{(0)} = W_{1/2}(0)$ is the unperturbed massless Dirac operator on half-densities (3.20). We denote by $\lambda^{(0)}$ the eigenvalue of this operator and by $v^{(0)}$ the corresponding eigenvector. Each eigenvalue $\lambda^{(0)}$ has even multiplicity because the massless Dirac operator commutes with the antilinear operator of charge conjugation (3.14).

The perturbation of an isolated eigenvalue of finite multiplicity of a bounded operator was described by Rellich [84] and that procedure can be applied in our case with some additional conditions.

The operators $W_{1/2}^{(k)}$, $k = 0, 1, 2, \ldots$ are formally self-adjoint first order differential operators which also commute with the antilinear operator of charge conjugation (3.14). We suppose that the series (3.39) is convergent for a sufficiently small ϵ. Then, see [84], there exist power series

$$\lambda(\epsilon) = \lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \cdots\,, \tag{3.40}$$

$$v(\epsilon) = v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots\,. \tag{3.41}$$

which are convergent for sufficiently small ϵ, which satisfy the condition

$$W_{1/2}(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon).$$

In the perturbation process that we use in this chapter we deal with a formally self-adjoint linear operator A which commutes with the antilinear operator of charge conjugation (3.14). Such an operator possess a special property stated by following lemma, see [24].

Lemma 3.4.3. *Let $A : C^\infty(M;\mathbb{C}^2) \to C^\infty(M;\mathbb{C}^2)$ be a (possibly unbounded) formally self-adjoint linear operator which commutes with the antilinear operator of charge conjugation* (3.14). *Then for any $v \in C^\infty(M;\mathbb{C}^2)$ we have*

$$\langle Av, C(v)\rangle = 0. \tag{3.42}$$

Proof. We prove of this lemma using the properties (3.17)-(3.19) of the charge conjugation operator (3.14). Let $v, w \in C^\infty(M;\mathbb{C}^2)$ be arbitrary elements. Then we have

$$\langle A(C(w)), C(v)\rangle = \langle C(A(w)), C(v)\rangle = \langle v, A(w)\rangle = \langle A(v), w\rangle. \tag{3.43}$$

For the $w = C(v)$ formula (3.43) reads

$$\langle A(C(C(v))), C(v)\rangle = \langle A(v), C(v)\rangle. \tag{3.44}$$

Since we have $C(C(v)) = -v$, the formula (3.44) becomes

$$-\langle A(v), C(v)\rangle = \langle A(v), C(v)\rangle,$$

which gives us (3.42). $\square$

We will use the *pseudoinverse* operator which we will define similarly to how it was done in [24, 84].

3.4.1 Pseudoinverse operator construction

Let $v^{(0)}$ be a normalised eigenvector of the operator A corresponding to the eigenvalue $\lambda^{(0)}$. Then, see Lemma 3.3.6, the vector $C(v^{(0)})$ is also a normalised eigenvector corresponding to the eigenvalue $\lambda^{(0)}$. Consider the problem

$$\left(A - \lambda^{(0)}\right) v = f, \tag{3.45}$$

for a given function $f \in L^2(M;\mathbb{C}^2)$ where we need to find the function $v \in H^1(M;\mathbb{C}^2)$. Suppose that the function f satisfies the conditions

$$\langle f, v^{(0)}\rangle = \langle f, C(v^{(0)})\rangle = 0,$$

where C is the charge conjugation operator (3.14). The problem (3.45) can be resolved for the function v and the uniqueness of this function is achieved by the conditions

$$\langle v, v^{(0)}\rangle = \langle v, C(v^{(0)})\rangle = 0.$$

We define the operator Q as

$$Q : f \mapsto v.$$

The operator Q is a bounded linear operator acting on the orthogonal complement of the eigenspace of the operator A corresponding to the eigenvalue $\lambda^{(0)}$. Also, the bounded linear operator Q is self-adjoint and commutes with the antilinear operator of charge conjugation (3.14). We can extend the acting of the pseudoinverse operator Q to the whole Hilbert space $L^2(M;\mathbb{C}^2)$ in accordance with $Qv^{(0)} = QC(v^{(0)}) = 0$.

Now we construct the pseudoinverse operator Q itself, as was explained in [84]. Let λ be an eigenvalue of multiplicity k of a Hermitian operator A. Then the homogenous equation

$$(A - \lambda^{(0)})v = 0$$

has k linearly independent solutions $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(k)}$ for which we can assume orthonormality, i.e.

$$\langle \phi^{(i)}, \phi^{(j)}\rangle = \delta_{ij}, \quad (i, j = 1, 2, \dots, k).$$

The operator $A-\lambda^{(0)}$ has no inverse, but there is a unique bounded Hermitian operator Q such that $Q\phi^{(i)}=0$, $(i=1,2,\ldots,k)$ and

$$Q(A-\lambda^{(0)})u = u - \sum_{i=1}^{k}\langle\phi^{(i)},u\rangle\phi^{(i)}.$$

Define the projector operator P by

$$Pu = \sum_{i=1}^{k}\langle\phi^{(i)},u\rangle\phi^{(i)}. \tag{3.46}$$

Then the properties of the above operator Q can be written as $QP=0$ and $Q(A-\lambda^{(0)})=I-P$.

The operator Q is called the pseudoinverse operator of the operator $A-\lambda$.

Now, we can complete the eigenvectors $\phi^{(1)},\phi^{(2)},\ldots,\phi^{(k)}$ with eigenvectors $\phi^{(k+1)},\phi^{(k+2)},\ldots,\phi^{(n)}$, which correspond to eigenvalues $\lambda_{k+1},\lambda_{k+2},\ldots,\lambda_n$, respectively, such that

$$\langle\phi^{(i)},\phi^{(j)}\rangle=\delta_{ij},\quad (i,j=1,2,\ldots,n).$$

Expanding v and f as

$$v=\sum_{i=1}^{n}\langle\phi^{(i)},v\rangle\phi^{(i)},\quad f=\sum_{i=1}^{n}\langle\phi^{(i)},f\rangle\phi^{(i)},$$

from equation (3.45) we get that

$$\begin{aligned}&\langle\phi^{(i)},f\rangle=0,\quad (i=1,2,\ldots,k),\\ &\langle\phi^{(i)},f\rangle=(\lambda_i-\lambda)\langle\phi^{(i)},v\rangle,\quad (i=k+1,k+2,\ldots,n).\end{aligned}$$

If the equation for v has a solution, it is necessary for f to be orthogonal to all solutions of the homogeneous equation. Hence we set

$$v=\sum_{i=1}^{k}v_i\phi^{(i)}+\sum_{\lambda_i\neq\lambda}\frac{\langle\phi^{(i)},f\rangle}{\lambda_i-\lambda}\phi^{(i)},$$

where v_i are arbitrary constants. The vector v defines the complete solution of the equation.

Let the operator P be the projector operator into the space spanned by the vectors $\phi^{(1)},\phi^{(2)},\ldots,\phi^{(k)}$, and P_i the projector into the one-dimensional space spanned by $\phi^{(i)}$, $(i=k+1,k+2,\ldots,n)$.

Definition 3.4.4. The operator

$$Q=\sum_{\lambda_i\neq\lambda^{(0)}}\frac{P_i}{\lambda_i-\lambda^{(0)}} \tag{3.47}$$

is the *pseudoinverse operator* of the operator $A-\lambda^{(0)}$.

3.4.2 Explicit formulae for the asymptotic coefficients

Now we will derive the explicit formulae for the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ in the asymptotic expansion (3.40). Consider the perturbed eigenvalue problem

$$W_{1/2}(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon).$$

Using formulae (3.39), (3.40) and (3.41), we get that

$$\begin{aligned}(W_{1/2}^{(0)} + \epsilon W_{1/2}^{(1)} + \epsilon^2 W_{1/2}^{(2)} + \cdots)(v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots) \\ = (\lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \cdots)(v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots).\end{aligned}$$

By grouping together the elements not containing ϵ, we get that

$$W_{1/2}^{(0)}v^{(0)} = \lambda^{(0)}v^{(0)},$$

which is the unperturbed eigenvalue problem. By grouping together the elements containing ϵ, we get that

$$W_{1/2}^{(0)}v^{(1)} + W_{1/2}^{(1)}v^{(0)} = \lambda^{(0)}v^{(1)} + \lambda^{(1)}v^{(0)}$$

and hence $(W_{1/2}^{(0)} - \lambda^{(0)})v^{(1)} = (\lambda^{(1)} - W_{1/2}^{(1)})v^{(0)}$, i.e. $v^{(1)} = Q(\lambda^{(1)} - W_{1/2}^{(1)})v^{(0)}$ where Q is the pseudoinverse operator of the operator $W_{1/2}^{(0)} - \lambda^{(0)}$. We denote by

$$f^{(1)} = (\lambda^{(1)} - W_{1/2}^{(1)})v^{(0)}. \tag{3.48}$$

By grouping together the elements containing ϵ^2, we get that

$$W_{1/2}^{(0)}v^{(2)} + W_{1/2}^{(1)}v^{(1)} + W_{1/2}^{(2)}v^{(0)} = \lambda^{(0)}v^{(2)} + \lambda^{(1)}v^{(1)} + \lambda^{(2)}v^{(0)}$$

and hence

$$(W_{1/2}^{(0)} - \lambda^{(0)})v^{(2)} = (\lambda^{(2)} - W_{1/2}^{(2)})v^{(0)} + (\lambda^{(1)} - W_{1/2}^{(1)})v^{(1)}.$$

We denote by

$$\begin{aligned}f^{(2)} &= (\lambda^{(2)} - W_{1/2}^{(2)})v^{(0)} + (\lambda^{(1)} - W_{1/2}^{(1)})v^{(1)} \\ &= (\lambda^{(2)} - W_{1/2}^{(2)})v^{(0)} + (\lambda^{(1)} - W_{1/2}^{(1)})Q(\lambda^{(1)} - W_{1/2}^{(1)})v^{(0)}.\end{aligned} \tag{3.49}$$

Continuing this process, the vectors $f^{(k)}$ and the coefficients $\lambda^{(k)}$ are obtained from the conditions

$$\langle f^{(k)}, v^{(0)}\rangle = 0 \tag{3.50}$$

and

$$\langle f^{(k)}, C(v^{(0)})\rangle = 0. \tag{3.51}$$

Remark 3.4.5. The eigenvalues have even multiplicity, so the condition (3.51) is an additional condition which need to be satisfied. This is the part where our perturbation process differs from the standard perturbation process for single eigenvalues.

The components $v^{(k)}$ are given by

$$v^{(k)} = Qf^{(k)},$$

where Q is the pseudoinverse operator of the operator $W^{(0)}_{1/2} - \lambda^{(0)}$. Substituting formulae (3.48), (3.49) into formulae (3.50), (3.51) we obtain the result stated by the following

Lemma 3.4.6. *The explicit formulae for the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ in the asymptotic expansion of the eigenvalue $\lambda(\epsilon)$ are given by*

$$\lambda^{(1)} = \langle W^{(1)}_{1/2} v^{(0)}, v^{(0)} \rangle, \tag{3.52}$$

$$\lambda^{(2)} = \langle W^{(2)}_{1/2} v^{(0)}, v^{(0)} \rangle - \langle (W^{(1)}_{1/2} - \lambda^{(1)}) Q (W^{(1)}_{1/2} - \lambda^{(1)}) v^{(0)}, v^{(0)} \rangle. \tag{3.53}$$

where $\langle\ \cdot\ ,\ \cdot\ \rangle$ is the inner product defined by (3.13).

3.5 Spectral asymmetry of the massless Dirac operator on a 3-torus

We now want to apply the results and the approach from the previous sections in this chapter in order to obtain and demonstrate spectral asymmetry of the massless Dirac operator (3.8) on a particular manifold. Hence, consider the unit tours $\mathbb{T}^3$ parameterised by cyclic coordinates x^α, $\alpha = 1, 2, 3$ of period 2π. For the Euclidean metric, the massless Dirac operator corresponding to the standard spin structure is given by the formula (3.29) and the eigenvalues can be evaluated explicitly. The spectrum is symmetric about zero and zero itself is an eigenvalue of the operator. But, as we stated before and according to [3, 4, 5, 6], for a general oriented Riemannian 3-manifold (M, g) there is no reason for the spectrum of the massless Dirac operator (3.8) to be symmetric.

We obtain the spectral asymmetry of the massless Dirac operator on a unit torus $\mathbb{T}^3$ by perturbing the Euclidean metric, as described in Section 3.4. By perturbing the metric we get the perturbed coframe (3.37) and frame (3.38), and hence the massless Dirac operator on half-densities (3.20) is also perturbed accordingly and we investigate under which perturbations of the metric the spectral symmetry of this operator is broken.

Definition 3.5.1. For a given function $f : \mathbb{T}^3 \to \mathbb{C}$, we denote by

$$\widehat{f}(m) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} e^{-\mathrm{i} m_\alpha x^\alpha} f(x) dx, \quad m \in \mathbb{Z}^3, \tag{3.54}$$

its Fourier coefficients. Here $dx := dx^1 dx^2 dx^3$.

Remark 3.5.2. An important special case that we will consider is when the metric is a function of the coordinate x^1 only. In this case one can choose the coframe and frame so that they depend on the coordinate x^1 only and seek eigenfunctions in the form $v(x^1)$. We call this case the *axisymmetric case*. In the above setting the original eigenvalue problem for a partial differential operator reduces to an eigenvalue problem for an ordinary differential operator.

Remark 3.5.3. For the Euclidean metric the massless Dirac operator (3.29) corresponding to the standard spin structure in the axisymmetric case reads

$$W = -\mathrm{i} \begin{pmatrix} 0 & \frac{d}{dx^1} \\ \frac{d}{dx^1} & 0 \end{pmatrix}. \tag{3.55}$$

A first order differential operator A is completely determined by its principal and subprincipal symbols, see Definition 3.1.1 and Definition 3.1.2. The principal symbol has the form $A_1(x, \xi) := M^{(\alpha)}(x)\xi_\alpha$ where $M^{(\alpha)}(x)$ are matrix-functions depending only on the position variable x and the differential operator A is given by

$$A = -\frac{\mathrm{i}}{2} M^{(\alpha)}(x) \frac{\partial}{\partial x^\alpha} - \frac{\mathrm{i}}{2} \frac{\partial}{\partial x^\alpha} M^{(\alpha)}(x) + A_{\mathrm{sub}}. \tag{3.56}$$

Let us now consider the axisymmetric case, see Remark 3.5.2. For $\alpha = 1$ in (3.56), using the formulae for the principal symbol (3.21) and subprincipal symbol (3.22), we straightforwardly get the explicit formula for the axisymmetric massless Dirac operator on half-densities in terms of the coframe and the frame

$$\begin{aligned} W_{1/2}(\epsilon) = &- \frac{\mathrm{i}}{2} \begin{pmatrix} e_3{}^1 & e_1{}^1 - \mathrm{i} e_2{}^1 \\ e_1{}^1 + \mathrm{i} e_2{}^1 & -e_3{}^1 \end{pmatrix} \frac{d}{dx^1} \\ &- \frac{\mathrm{i}}{2} \frac{d}{dx^1} \begin{pmatrix} e_3{}^1 & e_1{}^1 - \mathrm{i} e_2{}^1 \\ e_1{}^1 + \mathrm{i} e_2{}^1 & -e_3{}^1 \end{pmatrix} \\ &+ \frac{\delta_{jk}}{4\sqrt{\det g_{\alpha\beta}}} \left(e^j{}_3 \left(\frac{de^k{}_2}{dx^1} \right) - e^j{}_2 \left(\frac{de^k{}_3}{dx^1} \right) \right) I, \end{aligned} \tag{3.57}$$

corresponding to the perturbed metric $g(x^1, \epsilon)$ and where I is the 2×2 identity matrix and

$$\sqrt{\det g_{\alpha\beta}} = \frac{1}{\sqrt{\det g^{\alpha\beta}}} = \det e^j{}_\alpha = \frac{1}{\det e_j{}^\alpha}.$$

The coframe $e^j{}_\alpha(x^1; \epsilon)$ and the frame $e_j{}^\alpha(x^1; \epsilon)$ are defined in accordance with (3.37) and (3.38), respectively.

For an eigenvalue $n \in \mathbb{Z}$ the corresponding normalised eigenvector of the massless Dirac operator in the axisymmetric case (3.55) is

$$v_n(x^1) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{inx^1}. \tag{3.58}$$

Note that the vector

$$w_n(x^1) = C(v_n(x^1)) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-inx^1} \tag{3.59}$$

is also a normalised eigenvector of the operator (3.55) corresponding to eigenvalue $n \in \mathbb{Z}$, where C is the charge conjugation operator (3.14), since each eigenvalue of the massless Dirac operator in the axisymmetric case is of multiplicity two, see Lemma 3.3.6.

For the eigenvalue $\lambda = 0$ of the massless Dirac operator on half-densities the asymptotic formula (3.40) was obtained in the general case by Downes, Levitin and Vassiliev in [24], where the perturbations of the metric (3.36) with $k_{\alpha\beta}(x) = 0$ were considered. The linear coefficient $\lambda^{(1)}$ is equal to zero and the explicit formula for the quadratic coefficient $\lambda^{(2)}$ was obtained. The formula for the quadratic coefficient $\lambda^{(2)}$ was expressed via the components of perturbation matrix $h_{\alpha\beta}(x)$:

$$\lambda^{(2)} = \frac{\mathrm{i}}{16} \varepsilon_{\alpha\beta\gamma} \sum_{m \in \mathbb{Z}^3 \setminus \{0\}} \left(\delta_{\mu\nu} - \frac{m_\mu m_\nu}{\|m\|^2} \right) m_\alpha \widehat{h}_{\beta\mu}(m) \overline{\widehat{h}_{\gamma\nu}(m)}. \tag{3.60}$$

Here $\varepsilon_{\alpha\beta\gamma}$ is the totally antisymmetric quantity, $\varepsilon_{123} := +1$, and the overline stands for complex conjugation.

In the axisymmetric case, see Remark 3.5.2, formula (3.60) is simpler and reads

$$\lambda^{(2)} = -\frac{1}{8} \sum_{m_1 \in \mathbb{N}} m_1 \operatorname{tr} \left(\begin{pmatrix} \widehat{h}_{22} & \widehat{h}_{23} \\ \widehat{h}_{32} & \widehat{h}_{33} \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \widehat{h}_{22} & \widehat{h}_{23} \\ \widehat{h}_{32} & \widehat{h}_{33} \end{pmatrix}^* \right),$$

where $\widehat{h}_{\alpha\beta} = \widehat{h}_{\alpha\beta}(m_1)$ and '$*$' stands for Hermitian conjugation.

In [24] the authors also determined the conditions under which the constant $\lambda^{(2)}$ is nonzero, which tells us that for a sufficiently small nonzero ϵ the spectrum of our massless Dirac operator is asymmetric about zero. Also, the authors give two explicit examples of perturbation of the Euclidean metric, for which the eigenvalues of massless Dirac operator on half-densities in the axisymmetric case have been evaluated explicitly. One is an example of quadratic dependence and the second is an example of quartic dependence on the parameter ϵ.

In this chapter we take a similar approach to that taken in [24], but crucially we consider the eigenvalues ± 1 of the massless Dirac operator in the axisymmetric case (3.57) and for the perturbed Euclidean metric (3.36) we derive their asymptotic formulae of type (3.40), i.e. we seek the perturbations of the Euclidean metric for which it is possible to shift the eigenvalues ± 1 in an asymmetric way.

3.5.1 Numerical analysis of the spectrum

In this section we first numerically analyse the spectrum of the massless Dirac operator (3.55) using the *Galerkin method* to discretise the eigenvalue problem of the massless Dirac operator. We work on the unit torus $\mathbb{T}^3$ equipped with the Euclidean metric. Then for the standard spin structure we can calculate the spectrum of the massless Dirac operator (3.55) explicitly.

Consider the $2m+1$ eigenvalues $\lambda_i = i, \quad (i = 0, \pm 1, \ldots, \pm m)$ of the unperturbed massless Dirac operator on half-densities $W_{1/2}(0)$. Each eigenvalue λ_i $(i = 0, \pm 1, \ldots, \pm m)$ has multiplicity two and the corresponding eigenvectors $v_i(x^1)$ and $w_i(x^1)$ $(i = 0, \pm 1, \ldots, \pm m)$ are given by (3.58) and (3.59). Now, we have that

$$W_{1/2}(0) v_i(x^1) = \lambda_i v_i(x^1), \tag{3.61}$$
$$W_{1/2}(0) w_i(x^1) = \lambda_i w_i(x^1), \tag{3.62}$$

where $i = 0, \pm 1, \ldots, \pm m$. The eigenvectors $v_i(x^1)$ and $w_i(x^1)$ are orthonormal with respect to the inner product (3.13), i.e.

$$\begin{aligned} &\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = \delta_{ij}, \\ &\langle v_i, w_j \rangle = \langle w_i, v_j \rangle = 0, \ \ (i, j = 0, \pm 1, \ldots, \pm m). \end{aligned} \tag{3.63}$$

According to (3.63), from equations (3.61) and (3.62), for $i, j = 0, \pm 1, \ldots, \pm m$ we have that

$$\lambda_i = \langle W_{1/2}(0) v_i(x^1), v_i(x^1) \rangle = \langle W_{1/2}(0) w_i(x^1), w_i(x^1) \rangle$$

and

$$\langle W_{1/2}(0)v_i(x^1), w_j(x^1)\rangle = \langle W_{1/2}(0)v_j(x^1), w_i(x^1)\rangle = 0,$$
$$\langle W_{1/2}(0)w_i(x^1), v_j(x^1)\rangle = \langle W_{1/2}(0)w_j(x^1), v_i(x^1)\rangle = 0.$$

Let us now construct the matrices

$$H_{i,j} := \begin{pmatrix} \langle W_{1/2}(0)v_i, v_j\rangle & \langle W_{1/2}(0)v_i, w_j\rangle \\ \langle W_{1/2}(0)w_i, v_j\rangle & \langle W_{1/2}(0)w_i, w_j\rangle \end{pmatrix} \tag{3.64}$$

where $i, j = 0, \pm 1, \ldots, \pm m$. Using the matrices (3.64) let us construct the block matrix H as follows

$$H := \begin{pmatrix} H_{-m,m} & & & H_{0,m} & & & H_{m,m} \\ & \ddots & & \vdots & & \iddots & \\ & & & H_{0,1} & & & \\ & \cdots & H_{-1,0} & H_{0,0} & H_{1,0} & \cdots & \\ & & & H_{0,-1} & & & \\ & \iddots & & \vdots & & \ddots & \\ H_{-m,-m} & & & H_{0,-m} & & & H_{m,-m} \end{pmatrix}.$$

The matrix H is a quadratic matrix of order $2(2m+1)$ and by construction it is a Hermitian matrix.

The eigenvalues of the matrix H are $\lambda = 0, \pm 1, \ldots, \pm m$ and each eigenvalue has multiplicity two. It means that we reduced the eigenvalue problem (3.61)-(3.62) to the eigenvalue problem of the matrix H, using the Galerkin method.

Now we can consider the matrix $H(\epsilon)$ with the perturbed massless Dirac operator $W_{1/2}(\epsilon)$ instead of the unperturbed operator $W_{1/2}(0)$. Then the matrix $H(\epsilon)$ is a Hermitian matrix whose entries depend on the parameter ϵ and $H(0) = H$. Using the perturbation process described by McCartin [61] for perturbed Hermitian matrices, we can get the asymptotic expansions of the eigenvalues of the perturbed matrix $H(\epsilon)$, and specially the asymptotic expansions of the eigenvalues $\lambda = \pm 1$. The eigenvalues of the matrix $H(\epsilon)$ will clearly converge to the eigenvalues of the matrix H as $\epsilon \to 0$.

Remark 3.5.4. Throughout this chapter we denote by $\lambda_+(\epsilon)$ and $\lambda_-(\epsilon)$ the asymptotic expansions of the eigenvalues $\lambda = 1$ and $\lambda = -1$, respectively, and with $\lambda_+^{(i)}$ and $\lambda_-^{(i)}$ their asymptotic coefficients.

The examination of the spectrum of the perturbed massless Dirac operator will reduce to the examination of the spectrum of the Hermitian matrix $H(\epsilon)$.

Example 3.5.5. Consider the coframe

$$e^j{}_\alpha = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}. \tag{3.65}$$

The explicit formula for the perturbed massless Dirac operator corresponding to the coframe (3.65) was calculated in [24] and it reads

$$W(\epsilon) = -\mathrm{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx^1} - \frac{\epsilon^2}{2(1-\epsilon^2)} I. \tag{3.66}$$

The eigenvalues of the operator (3.66) are explicitly given by

$$\lambda_n(\epsilon) = n - \frac{\epsilon^2}{2(1-\epsilon^2)} = n - \frac{\epsilon^2}{2} - \frac{\epsilon^4}{2} + O(\epsilon^6), \quad n \in \mathbb{Z} \tag{3.67}$$

and all eigenvalues have multiplicity two.

Now we will use the coframe (3.65) to analyse the spectrum of the massless Dirac operator using the Galerkin method described at the beginning of this section in order to numerically confirm these results. We explicitly constructed the matrix $H(\epsilon)$ of order 102×102 and numerically analysed the part of its spectrum. The eigenvalues $0, \pm 1, \pm 2$ of the matrix $H(\epsilon)$ are perturbed as follows

	$\lambda = -2$	$\lambda = -1$	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$
for $\epsilon = 0.2$	-2.02083	-1.02083	-0.0208333	0.979167	1.97917
for $\epsilon = 0.1$	-2.00505	-1.00505	-0.00505051	0.994949	1.994950
for $\epsilon = 0.01$	-2.00005	-1.00005	-0.000050005	0.99995	1.99995

and each eigenvalue has multiplicity two. Analysing the data given in the above table we see that for this choice of the coframe the spectral symmetry of the matrix $H(\epsilon)$ is broken and consequently we obtain spectral asymmetry of the massless Dirac operator in the axisymmetric case.

Using the perturbation process for the matrices with double eigenvalues described in [61], we get that the asymptotic formulae for the eigenvalues ± 1 are given by

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3),$$
$$\lambda_-(\epsilon) = -1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3).$$

which is in accordance with formula (3.67).

Now, we will consider the coframe which is not symmetric to show that in this case it is also possible to obtain spectral asymmetry.

Example 3.5.6. Consider the coframe

$$e^j{}_\alpha = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & \cos x^1 - \cos 2x^1 + \cos 3x^1 & \sin x^1 + \sin 2x^1 - \sin 3x^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Analysing the spectrum of the matrix $H(\epsilon)$ of order 102×102 we get that the eigenvalues $0, \pm 1, \pm 2$ of the matrix are perturbed as follows

	$\lambda = -2$	$\lambda = -1$	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$
for $\epsilon = 0.2$	-2.10913	-1.05372	0.00169489	1.0571	2.11252
for $\epsilon = 0.1$	-2.02923	-1.01456	0.000119453	1.0148	2.02947
for $\epsilon = 0.01$	-2.0003	-1.00015	1.24941×10^{-8}	1.00015	2.0003

and each eigenvalue has multiplicity two. Analysing the data given in the above table we see that the spectral symmetry is broken.

Using the method described in [61], we obtain that the asymptotic formulae for the eigenvalues ± 1 are given by

$$\lambda_+(\epsilon) = 1 + \frac{3}{2}\epsilon^2 - \frac{17}{8}\epsilon^4 + O(\epsilon^5),$$
$$\lambda_-(\epsilon) = -1 - \frac{3}{2}\epsilon^2 + \frac{37}{8}\epsilon^4 + O(\epsilon^5),$$

and we see that spectral asymmetry is achieved in the quartic term.

3.5.2 Analytical formulae for perturbed eigenvalues

The numerical analysis of the spectrum of massless Dirac operator shows that for the appropriate choice of the coframe spectral asymmetry can be obtained. Now we want to prove this statement analytically, hence we derive explicitly the asymptotic formulae (3.40) for the eigenvalues ± 1 for an arbitrary perturbation of the Euclidean metric. The main result of this section is following

Theorem 3.5.7. *Under arbitrary perturbations of the metric* (3.36), *we have that*

$$\lambda_+(\epsilon) = 1 + \lambda_+^{(1)}\epsilon + \lambda_+^{(2)}\epsilon^2 + O(\epsilon^3) \quad as \quad \epsilon \to 0, \tag{3.68}$$
$$\lambda_-(\epsilon) = -1 + \lambda_-^{(1)}\epsilon + \lambda_-^{(2)}\epsilon^2 + O(\epsilon^3) \quad as \quad \epsilon \to 0, \tag{3.69}$$

where the constants $\lambda_+^{(1)}$, $\lambda_-^{(1)}$, $\lambda_+^{(2)}$ and $\lambda_-^{(2)}$ are given by the formulae

$$\lambda_+^{(1)} = -\frac{1}{2}\widehat{h}_{11}(0), \tag{3.70}$$

$$\lambda_-^{(1)} = \frac{1}{2}\widehat{h}_{11}(0), \tag{3.71}$$

$$\begin{aligned}\lambda_+^{(2)} = \frac{3}{8}\widehat{(h^2)}_{11}(0) - \frac{1}{8}\widehat{k}_{11}(0) - \frac{\mathrm{i}}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\ - \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1}(m+1)^2\ \widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)} \\ - \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{31}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right) \\ - \frac{\mathrm{i}}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{21}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right),\end{aligned} \tag{3.72}$$

$$\begin{aligned}\lambda_-^{(2)} = -\frac{3}{8}\widehat{(h^2)}_{11}(0) + \frac{1}{8}\widehat{k}_{11}(0) - \frac{\mathrm{i}}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\ - \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} \frac{1}{m+1}(m-1)^2\ \widehat{h}_{11}(m+1)\overline{\widehat{h}_{11}(m+1)} \\ - \frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{31}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right) \\ - \frac{\mathrm{i}}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{21}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right),\end{aligned} \tag{3.73}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the totally antisymmetric quantity, $\varepsilon_{123} := +1$, and the overline stands for complex conjugation.

Remark 3.5.8. We consider the massless Dirac operator on half-densities in the axisymmetric case (3.57), hence the metric $g_{\alpha\beta}$ depends on the coordinate x^1 only and for a given function h_{ij}, its Fourier coefficients appearing in Theorem 3.5.7 are given by

$$\widehat{h_{ij}}(m_1) := \frac{1}{2\pi}\int_0^{2\pi} e^{-im_1x^1} h_{ij}(x^1)dx^1, \quad m_1 \in \mathbb{Z}. \tag{3.74}$$

Theorem 3.5.7 warrants the following facts.

Remark 3.5.9. According to the formulae (3.70) and (3.71), we have that

$$\lambda_+^{(1)} + \lambda_-^{(1)} = 0,$$

so we conclude that the spectral asymmetry can't be achieved in the linear term.

Remark 3.5.10. The length of the arc of the x^1 circle is given by

$$l(\epsilon) = \int_0^{2\pi} \sqrt{g_{11}(x^1)}dx^1 = 2\pi\left(1 + \frac{1}{2}\widehat{h}_{11}(0)\epsilon\right) + O(\epsilon^2).$$

Hence the coefficients $\lambda_+^{(1)}$ and $\lambda_-^{(1)}$ are determined by the change of the length of the arc of x^1 circle.

Remark 3.5.11. According to formulae (3.72) and (3.73), we get that

$$\begin{aligned}
\lambda_+^{(2)} + \lambda_-^{(2)} = & -\frac{\mathrm{i}}{8}\varepsilon_{\beta\gamma 1} \sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1}(m+1)^2\,\widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)} \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{31}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right) \\
& -\frac{i}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\widehat{h}_{21}(m+1)\left(\overline{\widehat{h}_{31}(m+1)} - \mathrm{i}\overline{\widehat{h}_{21}(m+1)}\right) \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} \frac{1}{m+1}(m-1)^2\,\widehat{h}_{11}(m+1)\overline{\widehat{h}_{11}(m+1)} \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{31}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right) \\
& -\frac{i}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\widehat{h}_{21}(m-1)\left(\overline{\widehat{h}_{31}(m-1)} - \mathrm{i}\overline{\widehat{h}_{21}(m-1)}\right).
\end{aligned}$$

Hence, we conclude that one way to obtain spectral asymmetry is to choose such a perturbation matrix $h_{\alpha\beta}(x^1)$ so that $\widehat{h}_{11}(m) = \widehat{h}_{21}(m) = \widehat{h}_{31}(m) = 0$ for all $m \in \mathbb{Z}$ such that the term

$$\varepsilon_{\beta\gamma 1} \sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \neq 0.$$

In that case we have that $\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0$.

Proof of Theorem 3.5.7. In order to prove Theorem 3.5.7 we need to first write down explicitly the differential operators $W^{(1)}_{1/2}$ i $W^{(2)}_{1/2}$ appearing in the asymptotic expansion of the perturbed massless Dirac operator on half-densities

$$W_{1/2}(\epsilon) = W^{(0)}_{1/2} + \epsilon W^{(1)}_{1/2} + \epsilon^2 W^{(2)}_{1/2} + O(\epsilon^3).$$

Substituting formulae (3.37) and (3.38) into (3.57), we get that

$$\begin{aligned} W^{(1)}_{1/2} = & \frac{\mathrm{i}}{4}\begin{pmatrix} h_3{}^1 & h_1{}^1 - \mathrm{i}h_2{}^1 \\ h_1{}^1 + \mathrm{i}h_2{}^1 & -h_3{}^1 \end{pmatrix}\frac{d}{dx^1} \\ & + \frac{\mathrm{i}}{4}\frac{d}{dx^1}\begin{pmatrix} h_3{}^1 & h_1{}^1 - \mathrm{i}h_2{}^1 \\ h_1{}^1 + \mathrm{i}h_2{}^1 & -h_3{}^1 \end{pmatrix} \end{aligned} \tag{3.75}$$

and

$$\begin{aligned} W^{(2)}_{1/2} = & -\frac{3\mathrm{i}}{16}\begin{pmatrix} (h^2)_3{}^1 & (h^2)_1{}^1 - \mathrm{i}\,(h^2)_2{}^1 \\ (h^2)_1{}^1 + \mathrm{i}\,(h^2)_2{}^1 & -(h^2)_3{}^1 \end{pmatrix}\frac{d}{dx^1} \\ & - \frac{3\mathrm{i}}{16}\frac{d}{dx^1}\begin{pmatrix} (h^2)_3{}^1 & (h^2)_1{}^1 - \mathrm{i}\,(h^2)_2{}^1 \\ (h^2)_1{}^1 + \mathrm{i}\,(h^2)_2{}^1 & -(h^2)_3{}^1 \end{pmatrix} \\ & + \frac{\mathrm{i}}{16}\begin{pmatrix} k_3{}^1 & k_1{}^1 - \mathrm{i}k_2{}^1 \\ k_1{}^1 + \mathrm{i}k_2{}^1 & -k_3{}^1 \end{pmatrix}\frac{d}{dx^1} \\ & + \frac{\mathrm{i}}{16}\frac{d}{dx^1}\begin{pmatrix} k_3{}^1 & k_1{}^1 - \mathrm{i}k_2{}^1 \\ k_1{}^1 + \mathrm{i}k_2{}^1 & -k_3{}^1 \end{pmatrix} - \frac{1}{16}\varepsilon_{\beta\gamma 1}h_{\alpha\beta}\frac{dh_{\alpha\gamma}}{dx^1}I. \end{aligned} \tag{3.76}$$

The explicit formulae for the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ are given by (3.52) and (3.53).

Now consider the eigenvalue $\lambda = 1$ of the massless Dirac operator in axisymmetric case (3.57). The normalised eigenvector $v^{(0)}$ which corresponds to the eigenvalue $\lambda^{(0)} = 1$ is given by

$$v^{(0)}(x^1) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{ix^1}. \tag{3.77}$$

Of course, the vector

$$w^{(0)}(x^1) = C(v^{(0)}(x^1)) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-ix^1}$$

is also an eigenvector of the massless Dirac operator corresponding to the eigenvalue $n \in \mathbb{Z}$, where C is the charge conjugation operator (3.14), see formulae (3.58) and (3.59). We calculate the coefficients $\lambda^{(1)}_+$ and $\lambda^{(2)}_+$ for the eigenvector $v^{(0)}(x^1)$. Note that the result would be the same if we chose the eigenvector $w^{(0)}(x^1)$.

Using formulae (3.52), (3.75) for the eigenvector (3.77) and integrating by parts, we get that

$$\lambda_+^{(1)} = \langle W_{1/2}^{(1)} v^{(0)}, v^{(0)} \rangle = -\frac{1}{2}\widehat{h}_{11}(0),$$

see Appendix D for detailed calculations.

We will calculate the coefficient $\lambda_+^{(2)}$, see (3.53), in several stages, for sake of simplicity and readability. Let us first calculate the part $\langle W_{1/2}^{(2)} v^{(0)}, v^{(0)} \rangle$. Using (3.76), for the eigenvector (3.77) we get that

$$\langle W_{1/2}^{(2)} v^{(0)}, v^{(0)} \rangle = \frac{3}{8}\widehat{(h^2)}_{11}(0) - \frac{1}{8}\widehat{k}_{11}(0) - \frac{\mathrm{i}}{16}\varepsilon_{\beta\gamma 1} \sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m).$$

To calculate the part $\langle (W_{1/2}^{(1)} - \lambda^{(1)})Q(W_{1/2}^{(1)} - \lambda^{(1)})v^{(0)}, v^{(0)} \rangle$ we need the pseudoinverse operator Q. The construction of the pseudoinverse operator Q is explained in Section 3.4.1. For the eigenvectors (3.58) and (3.59) the projection operator (3.46) is given by

$$P = \frac{1}{4\pi}\sum_{m\in\mathbb{Z}}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{imx^1}\int_0^{2\pi}\begin{pmatrix} 1 & 1 \end{pmatrix}(\,\cdot\,)e^{-imy^1}dy^1 \right.$$
$$\left. + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-imx^1}\int_0^{2\pi}\begin{pmatrix} -1 & 1 \end{pmatrix}(\,\cdot\,)e^{imy^1}dy^1\right]$$

and hence using (3.47), the explicit formula for the pseudoinverse Q of the operator $W_{1/2}(0) - I$ is given by

$$Q = \frac{1}{4\pi}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\left[e^{imx^1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\int_0^{2\pi} e^{-imy^1}(\,\cdot\,)dy^1 \right.$$
$$\left. + e^{-imx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\int_0^{2\pi} e^{imy^1}(\,\cdot\,)dy^1\right] \tag{3.78}$$

Now, using (3.78), lengthy but straightforward calculations produce

$$\langle (W_{1/2}^{(1)} - \lambda^{(1)})Q(W_{1/2}^{(1)} - \lambda^{(1)})v^{(0)}, v^{(0)} \rangle =$$
$$\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}(m+1)^2\,\widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)} +$$
$$\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}}(m-1)\,\left(\widehat{h}_{31}(m+1) + i\widehat{h}_{21}(m+1)\right)\left(\overline{\widehat{h}_{31}(m+1)} - i\overline{\widehat{h}_{21}(m+1)}\right).$$

So, finally we get the formula (3.72).

The procedure of deriving of the formulae (3.71) and (3.73) is analogous to the derivations of the formulae (3.70) and (3.72). The only difference is that in calculations we use the eigenvector

$$v^{(0)}(x^1) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-ix^1}. \tag{3.79}$$

which correspond to the eigenvalue $\lambda = -1$ and the formula for the pseudoinverse Q of the operator $W_{1/2}(0) + I$ is given by

$$Q = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}\setminus\{-1\}} \frac{1}{m+1} \left[e^{imx^1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \int_0^{2\pi} e^{-imy^1}(\,\cdot\,)dy^1 \right.$$
$$\left. + e^{-imx^1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_0^{2\pi} e^{imy^1}(\,\cdot\,)dy^1 \right]. \tag{3.80}$$

Using formulae (3.52), (3.75) for the eigenvector (3.79) and integrating by parts, we get that

$$\lambda_-^{(1)} = \langle W_{1/2}^{(1)} v^{(0)}, v^{(0)} \rangle = \frac{1}{2}\widehat{h}_{11}(0).$$

We have that

$$\langle W_{1/2}^{(2)} v^{(0)}, v^{(0)} \rangle = -\frac{3}{8}\widehat{(h^2)}_{11}(0) + \frac{1}{8}\widehat{k}_{11}(0) - \frac{i}{16}\varepsilon_{\beta\gamma 1} \sum_{m \in \mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m)$$

and

$$\langle (W_{1/2}^{(1)} - \lambda^{(1)}) Q (W_{1/2}^{(1)} - \lambda^{(1)}) v^{(0)}, v^{(0)} \rangle =$$
$$\frac{1}{16} \sum_{m \in \mathbb{Z}\setminus\{-1\}} \frac{1}{m+1}(m-1)^2\, \widehat{h}_{11}(m+1)\overline{\widehat{h}_{11}(m+1)} +$$
$$\frac{1}{16} \sum_{m \in \mathbb{Z}\setminus\{-1\}} (m+1)\, \Big(\widehat{h}_{31}(m-1) + i\widehat{h}_{21}(m-1)\Big) \Big(\overline{\widehat{h}_{31}(m-1)} - i\overline{\widehat{h}_{21}(m-1)}\Big).$$

Combining the above results we obtain the formula (3.73). □

3.5.3 Explicit examples of spectral asymmetry

In this section we present two explicit examples of perturbation matrices for which the massless Dirac operator in the axisymmetric case (3.57) has an asymmetric spectrum. The explicit formulae for the massless Dirac operator are also derived for two different coframes.

The asymptotic formulae for the eigenvalues ± 1 are derived in two different ways, thus confirming our new results for the asymptotic coefficients from Theorem 3.5.7. First we derive the asymptotic coefficient of the eigenvalues ± 1 of the massless Dirac operator in the axisymmetric case by definition, using the formulae (3.52) and (3.53) and secondly, we use our new formulae (3.70), (3.71), (3.72) and (3.73) which are expressed only by using the Fourier's coefficients of the perturbation matrices h and k.

In the first example we present the perturbation matrices of the coframe for which the linear coefficients in the asymptotic expansions of the eigenvalues ± 1 are zero. In the second example, the linear factor is non-zero. These examples give us the guidelines on how to choose the perturbed coframe in order to obtain the spectral asymmetry of the massless Dirac operator.

Example 3.5.12. We choose the following perturbation matrices

$$h_{\alpha\beta}(x^1) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}, \quad k_{\alpha\beta}(x^1) = \begin{pmatrix} \sin x^1 & \cos x^1 & 0 \\ \cos x^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formulae (3.75) and (3.76), we get that

$$\begin{aligned} A^{(1)} &= 0, \\ A^{(2)} &= \frac{\mathrm{i}}{16}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1 + \sin x^1 \\ \mathrm{i}\cos x^1 + \sin x^1 & 0 \end{pmatrix}\frac{d}{dx^1} \\ &\quad + \frac{\mathrm{i}}{16}\frac{d}{dx^1}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1 + \sin x^1 \\ \mathrm{i}\cos x^1 + \sin x^1 & 0 \end{pmatrix} - \frac{1}{2}I, \end{aligned}$$

so the corresponding perturbed massless Dirac operator in the axisymmetric case (3.57) is explicitly given by

$$\begin{aligned} W(\epsilon) = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx^1} &+ \frac{\mathrm{i}\epsilon^2}{16}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1 + \sin x^1 \\ \mathrm{i}\cos x^1 + \sin x^1 & 0 \end{pmatrix}\frac{d}{dx^1} \\ &+ \frac{\mathrm{i}\epsilon^2}{16}\frac{d}{dx^1}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1 + \sin x^1 \\ \mathrm{i}\cos x^1 + \sin x^1 & 0 \end{pmatrix} - \frac{\epsilon^2}{2}I. \end{aligned}$$

Using formulae (3.13), (3.52), (3.53), (3.78) and (3.80) we get that

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3), \tag{3.81}$$

$$\lambda_-(\epsilon) = -1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3). \tag{3.82}$$

Application of the Fourier transform (3.74) gives us

$$\widehat{h}_{\alpha\beta}(m)=\begin{cases}\begin{pmatrix}0&0&0\\0&1&-\mathrm{i}\\0&-\mathrm{i}&-1\end{pmatrix} & \text{for}\quad m=1,\\ \\ \qquad 0 & \text{for}\quad m=2,3,\ldots,\end{cases}\tag{3.83}$$

$$\widehat{k}_{\alpha\beta}(m)=\begin{cases}\begin{pmatrix}-\mathrm{i}/2&1/2&0\\1/2&0&0\\0&0&0\end{pmatrix} & \text{for}\quad m=1,\\ \\ \qquad 0 & \text{for}\quad m=2,3,\ldots,\end{cases}\tag{3.84}$$

$$\widehat{(h^2)}_{\alpha\beta}(m)=\begin{cases}\begin{pmatrix}0&0&0\\0&4&0\\0&0&4\end{pmatrix} & \text{for}\quad m=0,\\ \\ \qquad 0 & \text{for}\quad m=1,2,3,\ldots\ .\end{cases}\tag{3.85}$$

Substituting (3.83), (3.84) and (3.85) into (3.70), (3.71), (3.72) and (3.73) we get that

$$\lambda^{(1)}_{\pm}=0,\ \lambda^{(2)}_{+}=\lambda^{(2)}_{-}=-\frac{1}{2}$$

which is in accordance with the asymptotic formulae (3.81) and (3.82).

Example 3.5.13. We choose the following perturbation matrices

$$h_{\alpha\beta}(x^1)=\begin{pmatrix}1&\cos x^1&\sin x^1\\ \cos x^1&\cos x^1&\sin x^1\\ \sin x^1&\sin x^1&-\cos x^1\end{pmatrix},\quad k_{\alpha\beta}(x^1)=\begin{pmatrix}\sin x^1&\cos x^1&0\\ \cos x^1&-\sin x^1&0\\0&0&0\end{pmatrix}.$$

Using formulae (3.75) and (3.76), we get that

$$A^{(1)}=\frac{\mathrm{i}}{4}\begin{pmatrix}\sin x^1&1-\mathrm{i}\cos x^1\\1+\mathrm{i}\cos x^1&-\sin x^1\end{pmatrix}\frac{d}{dx^1}+\frac{\mathrm{i}}{4}\frac{d}{dx^1}\begin{pmatrix}\sin x^1&1-\mathrm{i}\cos x^1\\1+\mathrm{i}\cos x^1&-\sin x^1\end{pmatrix}$$

$$\begin{aligned}A^{(2)}=&-\frac{3\mathrm{i}}{16}\begin{pmatrix}\sin x^1&2-\mathrm{i}-\mathrm{i}\cos x^1\\2+\mathrm{i}+\mathrm{i}\cos x^1&-\sin x^1\end{pmatrix}\frac{d}{dx^1}\\&-\frac{3\mathrm{i}}{16}\frac{d}{dx^1}\begin{pmatrix}\sin x^1&2-\mathrm{i}-\mathrm{i}\cos x^1\\2+\mathrm{i}+\mathrm{i}\cos x^1&-\sin x^1\end{pmatrix}\\&+\frac{\mathrm{i}}{16}\begin{pmatrix}0&-\mathrm{i}\cos x^1+\sin x^1\\ \mathrm{i}\cos x^1+\sin x^1&0\end{pmatrix}\frac{d}{dx^1}\\&+\frac{\mathrm{i}}{16}\frac{d}{dx^1}\begin{pmatrix}0&-\mathrm{i}\cos x^1+\sin x^1\\ \mathrm{i}\cos x^1+\sin x^1&0\end{pmatrix}-\frac{3}{16}I,\end{aligned}$$

so the corresponding perturbed massless Dirac operator in the axisymmetric case is given by

$$
\begin{aligned}
W(\epsilon) = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx^1} + \frac{\mathrm{i}\epsilon}{4}\begin{pmatrix} \sin x^1 & 1-\mathrm{i}\cos x^1 \\ 1+\mathrm{i}\cos x^1 & -\sin x^1 \end{pmatrix}\frac{d}{dx^1} \\
+\frac{\mathrm{i}\epsilon}{4}\frac{d}{dx^1}\begin{pmatrix} \sin x^1 & 1-\mathrm{i}\cos x^1 \\ 1+\mathrm{i}\cos x^1 & -\sin x^1 \end{pmatrix} \\
-\frac{3\mathrm{i}\epsilon^2}{16}\begin{pmatrix} \sin x^1 & 2-\mathrm{i}-\mathrm{i}\cos x^1 \\ 2+\mathrm{i}+\mathrm{i}\cos x^1 & -\sin x^1 \end{pmatrix}\frac{d}{dx^1} \\
-\frac{3\mathrm{i}\epsilon^2}{16}\frac{d}{dx^1}\begin{pmatrix} \sin x^1 & 2-\mathrm{i}-\mathrm{i}\cos x^1 \\ 2+\mathrm{i}+\mathrm{i}\cos x^1 & -\sin x^1 \end{pmatrix} \\
+\frac{\mathrm{i}\epsilon^2}{16}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1+\sin x^1 \\ \mathrm{i}\cos x^1+\sin x^1 & 0 \end{pmatrix}\frac{d}{dx^1} \\
+\frac{\mathrm{i}\epsilon^2}{16}\frac{d}{dx^1}\begin{pmatrix} 0 & -\mathrm{i}\cos x^1+\sin x^1 \\ \mathrm{i}\cos x^1+\sin x^1 & 0 \end{pmatrix} - \frac{3\epsilon^2}{16}I.
\end{aligned}
$$

Again using the formulae (3.13), (3.52), (3.53), (3.78) and (3.80) we get that

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon + \frac{3}{4}\epsilon^2 + O(\epsilon^3), \tag{3.86}$$

$$\lambda_-(\epsilon) = -1 + \frac{1}{2}\epsilon - \epsilon^2 + O(\epsilon^3). \tag{3.87}$$

Application of the Fourier transform (3.74) gives us

$$
\widehat{h}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for} \quad m = 0, \\ \\ \begin{pmatrix} 0 & 1/2 & -\mathrm{i}/2 \\ 1/2 & 1/2 & -\mathrm{i}/2 \\ -\mathrm{i}/2 & -\mathrm{i}/2 & -1/2 \end{pmatrix} & \text{for} \quad m = 1, \\ \\ 0 & \text{for} \quad m = 2, 3, \dots, \end{cases} \tag{3.88}
$$

$$
\widehat{k}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} -\mathrm{i}/2 & 1/2 & 0 \\ 1/2 & \mathrm{i}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for} \quad m = 1, \\ \\ 0 & \text{for} \quad m = 2, 3, \dots, \end{cases} \tag{3.89}
$$

$$\widehat{(h^2)}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} & \text{for} \quad m = 0, \\[2ex] \begin{pmatrix} 0 & 1/2 & -\mathrm{i}/2 \\ 1/2 & 0 & 0 \\ -\mathrm{i}/2 & 0 & 0 \end{pmatrix} & \text{for} \quad m = 1, \\[2ex] \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -\mathrm{i}/4 \\ 0 & -\mathrm{i}/4 & -1/4 \end{pmatrix} & \text{for} \quad m = 2, \\[2ex] 0 & \text{for} \quad m = 3, 4, \dots \, . \end{cases} \tag{3.90}$$

Substituting (3.88), (3.89) and (3.90) into (3.70), (3.71), (3.72) and (3.73) we get that

$$\lambda_{\pm}^{(1)} = \mp\frac{1}{2}, \quad \lambda_{+}^{(2)} = \frac{3}{4}, \text{ and } \lambda_{-}^{(2)} = -1,$$

which is in accordance with the asymptotic formulae (3.86) and (3.87).

Remark 3.5.14. In Example 3.5.12 we have shown that if we choose the perturbation matrix $h_{\alpha\beta}(x^1)$ such that $h_{11}(x^1) = 0$, the linear coefficients $\lambda_{\pm}^{(1)}$ are zero, which is in accordance with (3.70) and (3.71). However, if we choose the perturbation matrix $h_{\alpha\beta}(x^1)$ as in Example 3.5.13 with $h_{11}(x^1) \neq 0$, the linear coefficient is nonzero and for the eigenvalues ± 1 they have opposite sign. This is in accordance with Remark 3.5.9. The spectral asymmetry in both examples is obtained in the quadratic term.

Appendix A

Einstein and Yang-Mills Equations

In this appendix we present the derivation of the Einstein field equations, the Yang-Mills equation and the complementary Yang-Mills equation. These results are well known, see e.g. [53, 59, 72, 75, 99], but here we provide the detailed derivation of these equations.

A.1 The Einstein field equations

The Einstein field equations lie at the center of general relativity. It relates a spacetime geometry and a matter. The vacuum Einstein equations

$$Ric_{\alpha\beta} - \frac{1}{2}\mathcal{R}g_{\alpha\beta} = 0 \tag{A.1}$$

are obtained by varying the Einstein-Hilbert action

$$S_{EH} := \frac{c^4}{16\pi G}\int \mathcal{R}\sqrt{|\det g|}, \tag{A.2}$$

with respect to the metric g, as was previously stated in [75]. Here $\mathcal{R}$ is the scalar curvature (1.19), Ric the Ricci curvature (1.18), g is the metric, c is the speed of light and G is the gravitational constant, the recommended numerical value of which is $6.673\ 84(80) \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, with relative standard uncertainty 1.2×10^{-4}, see [63]. The full field equations are then obtained by adding the matter Lagrangian to the Einstein-Hilbert action, which gives us the Einstein equations in tensor form

$$Ric_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu},$$

where T is the stress energy tensor that arises from the matter Lagrangian, see e.g. [59]. The simplest solution of this equations is the Minkowski space-time from special relativity.

Proposition A.1.1. Let (M, g) be a (pseudo-)Riemannian manifold. Under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and $g_{\mu\nu}$, $\det g$ change as

- $\delta g^{\mu\nu} = -g^{\mu\kappa} g^{\lambda\nu} \delta g_{\kappa\lambda}$,
- $\delta \det g_{\mu\nu} = \det g_{\mu\nu} g^{\mu\nu} \delta g_{\mu\nu}$,
- $\delta \sqrt{|\det g_{\mu\nu}|} = \frac{1}{2} \sqrt{|\det g_{\mu\nu}|} g^{\mu\nu} \delta g_{\mu\nu}$.

For the proof of Proposition A.1.1 see [65]. Since $\mathcal{R} = R^{\kappa\mu}{}_{\kappa\mu}$, we have that

$$\begin{aligned}
\frac{\delta S_{EH}}{\delta g} &= \delta \int R^{\kappa\mu}{}_{\kappa\mu} \sqrt{|\det g|} = \delta \int R^{\kappa}{}_{\lambda\kappa\mu} g^{\lambda\mu} \sqrt{|\det g|} \\
&= \int \delta \left(R^{\kappa}{}_{\lambda\kappa\mu} g^{\lambda\mu} \sqrt{|\det g|} \right) = \int R^{\kappa}{}_{\lambda\kappa\mu} \delta \left(g^{\lambda\mu} \sqrt{|\det g|} \right) \\
&= \int R^{\kappa}{}_{\lambda\kappa\mu} \delta \left(-g^{\lambda\alpha} g^{\beta\mu} \delta g_{\alpha\beta} \sqrt{|\det g|} + g^{\lambda\mu} \frac{1}{2} \sqrt{|\det g|} g^{\alpha\beta} \delta g_{\alpha\beta} \right) \\
&= \int (\delta g_{\alpha\beta}) \left(-R^{\kappa}{}_{\lambda\kappa\mu} g^{\lambda\alpha} g^{\beta\mu} + \frac{1}{2} R^{\kappa}{}_{\lambda\kappa\mu} g^{\lambda\mu} g^{\alpha\beta} \right) \sqrt{|\det g|}.
\end{aligned}$$

So, we get that

$$\frac{\delta S_{EH}}{\delta g} = \int (\delta g_{\alpha\beta}) \left(-R^{\kappa\alpha}{}_{\kappa}{}^{\beta} + \frac{1}{2} \mathcal{R} g^{\alpha\beta} \right) = \int (\delta g_{\alpha\beta}) \left(-Ric^{\alpha\beta} + \frac{1}{2} \mathcal{R} g^{\alpha\beta} \right),$$

which gives us equation (A.1).

A.2 The Yang-Mills equations

We define the Yang-Mills action as

$$S_{YM} := \int R^{\kappa}{}_{\lambda\mu\nu} R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|}. \tag{A.3}$$

The variation of the action (A.3) with respect to the metric produces the so-called *complementary Yang-Mills equation* for the affine connection. Using

Proposition A.1.1, we get that

$$\begin{aligned}
\frac{\delta S_{YM}}{\delta g} &= \int \delta\left(R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa\mu'\nu'}g^{\mu\mu'}g^{\nu\nu'}\sqrt{|\det g|}\right)\\
&= \int (\delta g_{\alpha\beta})R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa\mu'\nu'}\left(-g^{\mu\alpha}g^{\beta\mu'}g^{\nu\nu'} - g^{\nu\alpha}g^{\beta\nu'}g^{\mu\mu'} + \frac{1}{2}g^{\mu\mu'}g^{\nu\nu'}g^{\alpha\beta}\right)\\
&= -\int (\delta g_{\alpha\beta})\left(R^{\kappa}{}_{\lambda\nu}{}^{\alpha}R^{\lambda}{}_{\kappa}{}^{\nu\beta} + R^{\kappa}{}_{\lambda\mu}{}^{\alpha}R^{\lambda}{}_{\kappa}{}^{\mu\beta} - \frac{1}{2}g^{\alpha\beta}R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu}\right)\\
&= -2\int (\delta g_{\alpha\beta})\left(R^{\kappa}{}_{\lambda\mu}{}^{\alpha}R^{\lambda}{}_{\kappa}{}^{\mu\beta} - \frac{1}{4}g^{\alpha\beta}R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu}\right), \qquad \text{(A.4)}
\end{aligned}$$

hence the complementary Yang-Mils equation is

$$R^{\kappa}{}_{\lambda\mu}{}^{\alpha}R^{\lambda}{}_{\kappa}{}^{\mu\beta} - \frac{1}{4}g^{\alpha\beta}R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu} = 0. \qquad \text{(A.5)}$$

Using the notation $H = H_{\alpha}{}^{\beta} := R^{\kappa}{}_{\lambda\mu\alpha}R^{\lambda}{}_{\kappa}{}^{\mu\beta}$ and $\delta = \delta_{\alpha}{}^{\beta}$ is the identity tensor, the equation (A.5) is equivalent to the equation

$$H - \frac{1}{4}(\mathrm{tr}\ H)\delta = 0,$$

as was obtained in [53].

By varying the action (A.3) with respect to the connection we get the *Yang - Mills equation* for the affine connection. As we vary the curvature independently, we have that

$$\begin{aligned}
\frac{\delta S_{YM}}{\delta \Gamma} &= \int \delta\left(R^{\kappa}{}_{\lambda\mu\nu}R^{\lambda}{}_{\kappa}{}^{\mu\nu}\right)\sqrt{|\det g|}\\
&= \int \delta\left(R^{\kappa}{}_{\lambda\mu\nu}\right)R^{\lambda}{}_{\kappa}{}^{\mu\nu}\sqrt{|\det g|} + \int \delta\left(R^{\lambda}{}_{\kappa}{}^{\mu\nu}\right)R^{\kappa}{}_{\lambda\mu\nu}\sqrt{|\det g|}\\
&= 2\int \delta\left(R^{\kappa}{}_{\lambda\mu\nu}\right)R^{\lambda}{}_{\kappa}{}^{\mu\nu}\sqrt{|\det g|}. \qquad \text{(A.6)}
\end{aligned}$$

Using the formula for the curvature tensor (1.17), we get that

$$\begin{aligned}
\delta R^{\kappa}{}_{\lambda\mu\nu} &= \partial_{\mu}\left(\delta\Gamma^{\kappa}{}_{\nu\lambda}\right) - \partial_{\nu}\left(\delta\Gamma^{\kappa}{}_{\mu\lambda}\right)\\
&\quad + \left(\delta\Gamma^{\kappa}{}_{\mu\eta}\right)\Gamma^{\eta}{}_{\nu\lambda} + \Gamma^{\kappa}{}_{\mu\eta}\delta\Gamma^{\eta}{}_{\nu\lambda} - \left(\delta\Gamma^{\kappa}{}_{\nu\eta}\right)\Gamma^{\eta}{}_{\mu\lambda} - \Gamma^{\kappa}{}_{\nu\eta}\delta\Gamma^{\eta}{}_{\mu\lambda}. \qquad \text{(A.7)}
\end{aligned}$$

Substituting equation (A.7) into equation (A.6), we get that

$$
\begin{aligned}
\frac{1}{2}\frac{\delta S_{YM}}{\delta \Gamma} &= -\int \left(\delta \Gamma^{\kappa}{}_{\nu\lambda}\right) \partial_{\mu} \left(R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\
&\quad + \int \left(\delta \Gamma^{\kappa}{}_{\mu\lambda}\right) \partial_{\nu} \left(R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\
&\quad + \int \left(\delta \Gamma^{\kappa}{}_{\mu\eta}\right) \Gamma^{\eta}{}_{\nu\lambda} R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} + \int \Gamma^{\kappa}{}_{\mu\eta} \left(\delta \Gamma^{\eta}{}_{\nu\lambda}\right) R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \\
&\quad - \int \left(\delta \Gamma^{\kappa}{}_{\nu\eta}\right) \Gamma^{\eta}{}_{\mu\lambda} R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} - \int \Gamma^{\kappa}{}_{\nu\eta} \left(\delta \Gamma^{\eta}{}_{\mu\lambda}\right) R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|}.
\end{aligned}
$$

Using the fact that for any curvature the antisymmetry $R^{\kappa}{}_{\lambda\mu\nu} = -R^{\kappa}{}_{\lambda\nu\mu}$ holds and renaming some indices, we get that

$$
\begin{aligned}
\frac{1}{2}\frac{\delta S_{YM}}{\delta \Gamma} &= 2\int \left(\delta \Gamma^{\kappa}{}_{\mu\lambda}\right) \partial_{\nu} \left(R^{\lambda}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} \right) \frac{\sqrt{|\det g|}}{\sqrt{|\det g|}} \\
&\quad + 2\int \left(\delta \Gamma^{\kappa}{}_{\mu\lambda}\right) \Gamma^{\lambda}{}_{\nu\eta} R^{\eta}{}_{\kappa}{}^{\mu\nu} \sqrt{|\det g|} - 2\int \Gamma^{\eta}{}_{\nu\kappa} \left(\delta \Gamma^{\kappa}{}_{\mu\lambda}\right) R^{\lambda}{}_{\eta}{}^{\mu\nu} \sqrt{|\det g|}.
\end{aligned}
$$

Then

$$
\frac{\delta S_{YM}}{\delta \Gamma} = 4\int \left(\delta \Gamma_{\mu}\right) \frac{1}{\sqrt{|\det g|}} \left(\partial_{\nu} + [\Gamma_{\nu}, \cdot]\right) \left(\sqrt{|\det g|} R^{\mu\nu} \right).
$$

Hence, the Yang-Mills equation is

$$
\left(\partial_{\nu} + [\Gamma_{\nu}\,,\,\cdot\,]\right) \left(\sqrt{|\det g|}\, R^{\mu\nu} \right) = 0.
$$

Appendix B

Bianchi Identity for Curvature

In this appendix we present the derivation of the Bianchi identity for curvature which is used in the derivation of the explicit form of the second field equation (2.50). We will use the following assumptions:

(i) Our spacetime is metric compatible;

(ii) Torsion is purely axial;

(iii) Ricci curvature is symmetric;

(iv) Scalar curvature $\mathcal{R}$ and pseudoscalar curvature $\mathcal{R}_*$ are zero.

Remark B.0.1. The antisymmetry $R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu}$ is true for any curvature and the antisymmetry $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$ is a consequence of the metric compatibility. Note that we will *not* use the symmetry $R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}$, since the curvature of generalised pp-waves with purely axial torsion does not possess this property.

B.1 Explicit formula for $R^{(5)}$ piece of curvature

Now we will derive the explicit formula for the $R^{(5)}$ irreducible piece of curvature (1.35), see Section 1.4.1 and [99] for more on this irreducible decomposition. It is known that the piece $R^{(5)}$ and the pieces $R_*^{(9,l)}$, see (1.41), are related as

$$R^{(5)} = -(R_*^{(9,1)} + R_*^{(9,2)})^*, \tag{B.1}$$

where $*$ denotes the right Hodge star (1.23). Using equation (B.1), we have that

$$
\begin{aligned}
R^{(5)}_{\kappa\lambda\mu\nu} = &-\Bigg(\frac{3}{8}(g_{\kappa\mu}{\mathcal{S}_*}^{(1)}{}_{\lambda\nu} - g_{\kappa\nu}{\mathcal{S}_*}^{(1)}{}_{\lambda\mu}) - \frac{1}{8}(g_{\lambda\mu}{\mathcal{S}_*}^{(1)}{}_{\kappa\nu} - g_{\lambda\nu}{\mathcal{S}_*}^{(1)}{}_{\kappa\mu}) \\
&- \frac{1}{8}(g_{\kappa\mu}{\mathcal{S}_*}^{(2)}{}_{\lambda\nu} - g_{\kappa\nu}{\mathcal{S}_*}^{(2)}{}_{\lambda\mu}) + \frac{3}{8}(g_{\lambda\mu}{\mathcal{S}_*}^{(2)}{}_{\kappa\nu} - g_{\lambda\nu}{\mathcal{S}_*}^{(2)}{}_{\kappa\mu})\Bigg)^* \\
= &\frac{1}{8}\left(\left(3{\mathcal{S}_*}^{(1)}{}_{\lambda\eta} - {\mathcal{S}_*}^{(2)}{}_{\lambda\eta}\right)\epsilon^\eta{}_{\kappa\mu\nu} + \left(3{\mathcal{S}_*}^{(2)}{}_{\kappa\eta} - {\mathcal{S}_*}^{(1)}{}_{\kappa\eta}\right)\epsilon^\eta{}_{\lambda\mu\nu}\right).
\end{aligned}
$$

For metric compatible spacetimes we have that $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$ and consequently ${\mathcal{S}^{(1)}_*}_{\mu\nu} = -{\mathcal{S}^{(2)}_*}_{\mu\nu}$. Hence, we get that

$$
R^{(5)}_{\kappa\lambda\mu\nu} = \frac{1}{4}\left((\mathcal{R}ic^{(1)}_*{}_{\lambda\eta} + \mathcal{R}ic^{(1)}_*{}_{\eta\lambda})\epsilon^\eta{}_{\kappa\mu\nu} - (\mathcal{R}ic^{(1)}_*{}_{\kappa\eta} + \mathcal{R}ic^{(1)}_*{}_{\eta\kappa})\epsilon^\eta{}_{\lambda\mu\nu}\right). \tag{B.2}
$$

Under the assumptions that Ric is symmetric and pseudoscalar curvature $\mathcal{R}_*$ is zero, we have the simplified version of formula (B.2)

$$
R^{(5)}_{\kappa\lambda\mu\nu} = \frac{1}{2}\left(\epsilon^\eta{}_{\kappa\mu\nu} Ric^{(1)}_*{}_{\lambda\eta} - \epsilon^\eta{}_{\lambda\mu\nu} Ric^{(1)}_*{}_{\kappa\eta}\right). \tag{B.3}
$$

B.2 The derivation of the Bianchi identity

Assumption (i) implies that the piece of curvature $R^{(6)}$ (1.36) is zero and assumption (iv) clearly implies that the $R^{(2)}$ (1.32) and $R^{(4)}$ (1.34) pieces of curvature are zero. Hence, under the above assumptions, the curvature (1.17) has only *three* nonzero irreducible pieces, namely $R^{(1)}$, $R^{(3)}$ and $R^{(5)}$. It can therefore be represented as

$$
\begin{aligned}
R_{\kappa\lambda\mu\nu} = &\frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} + g_{\lambda\nu}Ric_{\kappa\mu} - g_{\kappa\nu}Ric_{\lambda\mu}) + \mathcal{W}_{\kappa\lambda\mu\nu} \\
&+ \frac{1}{2}(-\epsilon^\eta{}_{\lambda\mu\nu}Ric_{*\kappa\eta} + \epsilon^\eta{}_{\kappa\mu\nu}Ric_{*\lambda\eta}).
\end{aligned} \tag{B.4}
$$

The Bianchi identity for curvature is

$$
\nabla_\xi R^\kappa{}_{\lambda\mu\nu} + \nabla_\nu R^\kappa{}_{\lambda\xi\mu} + \nabla_\mu R^\kappa{}_{\lambda\nu\xi} = 0,
$$

which can be written as

$$
(\partial_\xi + [\Gamma_\xi, \cdot])R_{\mu\nu} + (\partial_\nu + [\Gamma_\nu, \cdot])R_{\xi\mu} + (\partial_\mu + [\Gamma_\mu, \cdot])R_{\nu\xi} = 0, \tag{B.5}
$$

where we hide the Lie algebra indices of curvature by using matrix notation

$$
[\Gamma_\xi, R_{\mu\nu}]^\kappa{}_\lambda = \Gamma^\kappa{}_{\xi\eta}R^\eta{}_{\lambda\mu\nu} - R^\kappa{}_{\eta\mu\nu}\Gamma^\eta{}_{\xi\lambda}.
$$

Substituting (B.4) into (B.5) we get that

$$
\begin{aligned}
0 &= \partial_\xi \left[\frac{1}{2}(\delta^\kappa{}_\mu Ric_{\lambda\nu} - g_{\lambda\mu} Ric^\kappa{}_\nu + g_{\lambda\nu} Ric^\kappa{}_\mu - \delta^\kappa{}_\nu Ric_{\lambda\mu}) + \mathcal{W}^\kappa{}_{\lambda\mu\nu}\right] \\
&+ \frac{1}{2}\partial_\xi(-\epsilon^\vartheta{}_{\lambda\mu\nu} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\mu\nu} Ric_{*\lambda\vartheta}) \\
&+ \partial_\nu \left[\frac{1}{2}(\delta^\kappa{}_\xi Ric_{\lambda\mu} - g_{\lambda\xi} Ric^\kappa{}_\mu + g_{\lambda\mu} Ric^\kappa{}_\xi - \delta^\kappa{}_\mu Ric_{\lambda\xi}) + \mathcal{W}^\kappa{}_{\lambda\xi\mu}\right] \\
&+ \frac{1}{2}\partial_\nu(-\epsilon^\vartheta{}_{\lambda\xi\mu} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\xi\mu} Ric_{*\lambda\vartheta}) \\
&+ \partial_\mu \left[\frac{1}{2}(\delta^\kappa{}_\nu Ric_{\lambda\xi} - g_{\lambda\nu} Ric^\kappa{}_\xi + g_{\lambda\xi} Ric^\kappa{}_\nu - \delta^\kappa{}_\xi Ric_{\lambda\nu}) + \mathcal{W}^\kappa{}_{\lambda\nu\xi}\right] \\
&+ \frac{1}{2}\partial_\mu(-\epsilon^\vartheta{}_{\lambda\nu\xi} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\nu\xi} Ric_{*\lambda\vartheta}) \\
&+ \Gamma^\kappa{}_{\nu\eta} \left[\frac{1}{2}(\delta^\eta{}_\xi Ric_{\lambda\mu} - g_{\lambda\xi} Ric^\eta{}_\mu + g_{\lambda\mu} Ric^\eta{}_\xi - \delta^\eta{}_\mu Ric_{\lambda\xi}) + \mathcal{W}^\eta{}_{\lambda\xi\mu}\right] \\
&+ \frac{1}{2}\Gamma^\kappa{}_{\nu\eta}(-\epsilon^\vartheta{}_{\lambda\xi\mu} Ric_*{}^\eta{}_\vartheta + \epsilon^{\vartheta\eta}{}_{\xi\mu} Ric_{*\lambda\vartheta}) \\
&- \Gamma^\eta{}_{\nu\lambda} \left[\frac{1}{2}(\delta^\kappa{}_\xi Ric_{\eta\mu} - g_{\eta\xi} Ric^\kappa{}_\mu + g_{\eta\mu} Ric^\kappa{}_\xi - \delta^\kappa{}_\mu Ric_{\eta\xi}) + \mathcal{W}^\kappa{}_{\eta\xi\mu}\right] \\
&- \frac{1}{2}\Gamma^\eta{}_{\nu\lambda}(-\epsilon^\vartheta{}_{\eta\xi\mu} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\xi\mu} Ric_{*\eta\vartheta}) \\
&+ \Gamma^\kappa{}_{\xi\eta} \left[\frac{1}{2}(\delta^\eta{}_\mu Ric_{\lambda\nu} - g_{\lambda\mu} Ric^\eta{}_\nu + g_{\lambda\nu} Ric^\eta{}_\mu - \delta^\eta{}_\nu Ric_{\lambda\mu}) + \mathcal{W}^\eta{}_{\lambda\mu\nu}\right] \\
&+ \frac{1}{2}\Gamma^\kappa{}_{\xi\eta}(-\epsilon^\vartheta{}_{\lambda\mu\nu} Ric_*{}^\eta{}_\vartheta + \epsilon^{\vartheta\eta}{}_{\mu\nu} Ric_{*\lambda\vartheta}) \\
&- \Gamma^\eta{}_{\xi\lambda} \left[\frac{1}{2}(\delta^\kappa{}_\mu Ric_{\eta\nu} - g_{\eta\mu} Ric^\kappa{}_\nu + g_{\eta\nu} Ric^\kappa{}_\mu - \delta^\kappa{}_\nu Ric_{\eta\mu}) + \mathcal{W}^\kappa{}_{\eta\mu\nu}\right] \\
&- \frac{1}{2}\Gamma^\eta{}_{\xi\lambda}(-\epsilon^\vartheta{}_{\eta\mu\nu} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\mu\nu} Ric_{*\eta\vartheta}) \\
&+ \Gamma^\kappa{}_{\mu\eta} \left[\frac{1}{2}(\delta^\eta{}_\nu Ric_{\lambda\xi} - g_{\lambda\nu} Ric^\eta{}_\xi + g_{\lambda\xi} Ric^\eta{}_\nu - \delta^\eta{}_\xi Ric_{\lambda\nu}) + \mathcal{W}^\eta{}_{\lambda\nu\xi}\right] \\
&+ \frac{1}{2}\Gamma^\kappa{}_{\mu\eta}(-\epsilon^\vartheta{}_{\lambda\nu\xi} Ric_*{}^\eta{}_\vartheta + \epsilon^{\vartheta\eta}{}_{\nu\xi} Ric_{*\lambda\vartheta}) \\
&- \Gamma^\eta{}_{\mu\lambda} \left[\frac{1}{2}(\delta^\kappa{}_\nu Ric_{\eta\xi} - g_{\eta\nu} Ric^\kappa{}_\xi + g_{\eta\xi} Ric^\kappa{}_\nu - \delta^\kappa{}_\xi Ric_{\eta\nu}) + \mathcal{W}^\kappa{}_{\eta\nu\xi}\right] \\
&- \frac{1}{2}\Gamma^\eta{}_{\mu\lambda}(-\epsilon^\vartheta{}_{\eta\nu\xi} Ric_*{}^\kappa{}_\vartheta + \epsilon^{\vartheta\kappa}{}_{\nu\xi} Ric_{*\eta\vartheta}).
\end{aligned}
$$

A lengthy but straightforward calculation yields

$$
\begin{aligned}
0 = {} & \frac{1}{2}\,\{\delta^{\kappa}{}_{\mu}(\nabla_{\xi}Ric_{\lambda\nu} + \Gamma^{\eta}{}_{\xi\nu}Ric_{\lambda\eta}) + g_{\lambda\mu}(-\nabla_{\xi}Ric^{\kappa}{}_{\nu} - \Gamma^{\eta}{}_{\xi\nu}Ric^{\kappa}{}_{\eta}) \\
& + g_{\lambda\nu}(\nabla_{\xi}Ric^{\kappa}{}_{\mu} + \Gamma^{\eta}{}_{\xi\mu}Ric^{\kappa}{}_{\eta}) + \delta^{\kappa}{}_{\nu}(-\nabla_{\xi}Ric_{\lambda\mu} - \Gamma^{\eta}{}_{\xi\mu}Ric_{\lambda\eta}) \\
& + \delta^{\kappa}{}_{\xi}(\nabla_{\nu}Ric_{\lambda\mu} + \Gamma^{\eta}{}_{\nu\mu}Ric_{\lambda\eta}) + \delta^{\kappa}{}_{\mu}(-\nabla_{\nu}Ric_{\lambda\xi} - \Gamma^{\eta}{}_{\nu\xi}Ric_{\lambda\eta}) \\
& + g_{\lambda\mu}(\nabla_{\nu}Ric^{\kappa}{}_{\xi} + \Gamma^{\eta}{}_{\nu\xi}Ric^{\kappa}{}_{\eta}) + g_{\lambda\xi}(-\nabla_{\nu}Ric^{\kappa}{}_{\mu} - \Gamma^{\eta}{}_{\nu\mu}Ric^{\kappa}{}_{\eta}) \\
& + \delta^{\kappa}{}_{\nu}(\nabla_{\mu}Ric_{\lambda\xi} + \Gamma^{\eta}{}_{\mu\xi}Ric_{\lambda\eta}) + \delta^{\kappa}{}_{\xi}(-\nabla_{\mu}Ric_{\lambda\nu} - \Gamma^{\eta}{}_{\mu\nu}Ric_{\lambda\eta}) \\
& + g_{\lambda\xi}(\nabla_{\mu}Ric^{\kappa}{}_{\nu} + \Gamma^{\eta}{}_{\mu\nu}Ric^{\kappa}{}_{\eta}) + g_{\lambda\nu}(-\nabla_{\mu}Ric^{\kappa}{}_{\xi} - \Gamma^{\eta}{}_{\mu\xi}Ric^{\kappa}{}_{\eta}) \\
& + \Gamma^{\kappa}{}_{\nu\eta}(\delta^{\eta}{}_{\xi}Ric_{\lambda\mu} - \delta^{\eta}{}_{\mu}Ric_{\lambda\xi}) + \Gamma^{\eta}{}_{\nu\lambda}(g_{\eta\xi}Ric^{\kappa}{}_{\mu} - g_{\eta\mu}Ric^{\kappa}{}_{\xi}) \\
& + \Gamma^{\kappa}{}_{\xi\eta}(\delta^{\eta}{}_{\mu}Ric_{\lambda\nu} - \delta^{\eta}{}_{\nu}Ric_{\lambda\mu}) + \Gamma^{\eta}{}_{\xi\lambda}(g_{\eta\mu}Ric^{\kappa}{}_{\nu} - g_{\eta\nu}Ric^{\kappa}{}_{\mu}) \\
& + \Gamma^{\kappa}{}_{\mu\eta}(\delta^{\eta}{}_{\nu}Ric_{\lambda\xi} - \delta^{\eta}{}_{\xi}Ric_{\lambda\nu}) + \Gamma^{\eta}{}_{\mu\lambda}(g_{\eta\nu}Ric^{\kappa}{}_{\xi} - g_{\eta\xi}Ric^{\kappa}{}_{\nu}) \\
& + \partial_{\xi}\left[(-\epsilon^{\vartheta}{}_{\lambda\mu\nu}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\mu\nu}Ric_{*\lambda\vartheta})\right] + \partial_{\nu}\left[(-\epsilon^{\vartheta}{}_{\lambda\xi\mu}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\xi\mu}Ric_{*\lambda\vartheta})\right] \\
& + \partial_{\mu}\left[(-\epsilon^{\vartheta}{}_{\lambda\nu\xi}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\nu\xi}Ric_{*\lambda\vartheta})\right] + \Gamma^{\kappa}{}_{\nu\eta}\left[(-\epsilon^{\vartheta}{}_{\lambda\xi\mu}Ric_{*}{}^{\eta}{}_{\vartheta} + \epsilon^{\vartheta\eta}{}_{\xi\mu}Ric_{*\lambda\vartheta})\right] \\
& - \Gamma^{\eta}{}_{\nu\lambda}\left[(-\epsilon^{\vartheta}{}_{\eta\xi\mu}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\xi\mu}Ric_{*\eta\vartheta})\right] + \Gamma^{\kappa}{}_{\xi\eta}\left[(-\epsilon^{\vartheta}{}_{\lambda\mu\nu}Ric_{*}{}^{\eta}{}_{\vartheta} + \epsilon^{\vartheta\eta}{}_{\mu\nu}Ric_{*\lambda\vartheta})\right] \\
& - \Gamma^{\eta}{}_{\xi\lambda}\left[(-\epsilon^{\vartheta}{}_{\eta\mu\nu}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\mu\nu}Ric_{*\eta\vartheta})\right] + \Gamma^{\kappa}{}_{\mu\eta}\left[(-\epsilon^{\vartheta}{}_{\lambda\nu\xi}Ric_{*}{}^{\eta}{}_{\vartheta} + \epsilon^{\vartheta\eta}{}_{\nu\xi}Ric_{*\lambda\vartheta})\right] \\
& -\Gamma^{\eta}{}_{\mu\lambda}\left[(-\epsilon^{\vartheta}{}_{\eta\nu\xi}Ric_{*}{}^{\kappa}{}_{\vartheta} + \epsilon^{\vartheta\kappa}{}_{\nu\xi}Ric_{*\eta\vartheta})\right]\} \\
& + \partial_{\xi}\mathcal{W}^{\kappa}{}_{\lambda\mu\nu} + \partial_{\nu}\mathcal{W}^{\kappa}{}_{\lambda\xi\mu} + \partial_{\mu}\mathcal{W}^{\kappa}{}_{\lambda\nu\xi} + \Gamma^{\kappa}{}_{\nu\eta}\mathcal{W}^{\eta}{}_{\lambda\xi\mu} - \Gamma^{\eta}{}_{\nu\lambda}\mathcal{W}^{\kappa}{}_{\eta\xi\mu} \\
& + \Gamma^{\kappa}{}_{\xi\eta}\mathcal{W}^{\eta}{}_{\lambda\mu\nu} - \Gamma^{\eta}{}_{\xi\lambda}\mathcal{W}^{\kappa}{}_{\eta\mu\nu} + \Gamma^{\kappa}{}_{\mu\eta}\mathcal{W}^{\eta}{}_{\lambda\nu\xi} - \Gamma^{\eta}{}_{\mu\lambda}\mathcal{W}^{\kappa}{}_{\eta\nu\xi}.
\end{aligned}
$$

We will now contract the indices κ and μ in the above equation.

Since the above equation is too long, first we consider only the terms involving the Weyl curvature $\mathcal{W}$. Contracting the indices κ and μ and using the formula for the covariant derivative of Weyl tensor, we get that

$$
\begin{aligned}
& \nabla_{\mu}\mathcal{W}^{\mu}{}_{\lambda\nu\xi}\Gamma^{\mu}{}_{\nu\eta}\mathcal{W}^{\eta}{}_{\lambda\xi\mu} + \Gamma^{\mu}{}_{\xi\eta}\mathcal{W}^{\eta}{}_{\lambda\mu\nu} + \Gamma^{\eta}{}_{\mu\nu}\mathcal{W}^{\mu}{}_{\lambda\eta\xi} + \Gamma^{\eta}{}_{\mu\xi}\mathcal{W}^{\mu}{}_{\lambda\nu\eta} \\
& = \nabla_{\mu}\mathcal{W}^{\mu}{}_{\lambda\nu\xi} + \mathcal{W}^{\mu}{}_{\lambda\xi\eta}(\Gamma^{\eta}{}_{\nu\mu} - \Gamma^{\eta}{}_{\mu\nu}) + \mathcal{W}^{\mu}{}_{\lambda\nu\eta}(\Gamma^{\eta}{}_{\mu\xi} - \Gamma^{\eta}{}_{\xi\mu}) \\
& = \nabla_{\mu}\mathcal{W}^{\mu}{}_{\lambda\nu\xi} + \mathcal{W}^{\mu}{}_{\lambda\xi\eta}(K^{\eta}{}_{\nu\mu} - K^{\eta}{}_{\mu\nu}) + \mathcal{W}^{\mu}{}_{\lambda\nu\eta}(K^{\eta}{}_{\mu\xi} - K^{\eta}{}_{\xi\mu}),
\end{aligned}
$$

by the symmetry property of the Levi-Civita connection and equation (1.14).

Now we deal with the terms involving the covariant derivative of Ricci curvature, which after the contraction of the indices κ and μ become

$$
\begin{aligned}
& 4\nabla_{\xi}Ric_{\lambda\nu} - \nabla_{\xi}Ric_{\lambda\nu} - \nabla_{\xi}Ric_{\lambda\nu} + \nabla_{\nu}Ric_{\lambda\xi} - 4\nabla_{\nu}Ric_{\lambda\xi} + \nabla_{\nu}Ric_{\lambda\xi} \\
& + \nabla_{\nu}Ric_{\lambda\xi} - \nabla_{\xi}Ric_{\lambda\nu} + g_{\lambda\xi}\nabla_{\mu}Ric^{\mu}{}_{\nu} - g_{\lambda\nu}\nabla_{\mu}Ric^{\mu}{}_{\xi} \\
& = \nabla_{\xi}Ric_{\lambda\nu} - \nabla_{\nu}Ric_{\lambda\xi} + g_{\lambda\xi}\nabla_{\mu}Ric^{\mu}{}_{\nu} - g_{\lambda\nu}\nabla_{\mu}Ric^{\mu}{}_{\xi}.
\end{aligned}
$$

The other terms from the Bianchi identity involving Ricci tensor terms, after the contraction of the indices κ and μ and using the symmetry property of the Levi-Civita connection and equation (1.14) become

$$\begin{aligned}
&\frac{1}{2}[4Ric_{\lambda\eta}(K^{\eta}{}_{\xi\nu}-K^{\eta}{}_{\nu\xi})+Ric_{\lambda\eta}(K^{\eta}{}_{\nu\xi}-K^{\eta}{}_{\xi\nu})+g_{\lambda\nu}Ric^{\mu}{}_{\eta}(K^{\eta}{}_{\xi\mu}-K^{\eta}{}_{\mu\xi})\\
&\quad+Ric_{\lambda\eta}(K^{\eta}{}_{\nu\xi}-K^{\eta}{}_{\xi\nu})+Ric_{\lambda\eta}(K^{\eta}{}_{\nu\xi}-K^{\eta}{}_{\xi\nu})+g_{\lambda\xi}Ric^{\mu}{}_{\eta}(K^{\eta}{}_{\mu\nu}-K^{\eta}{}_{\nu\mu})\\
&\quad+Ric_{\lambda\mu}(K^{\mu}{}_{\nu\xi}-K^{\mu}{}_{\xi\nu})+Ric_{\lambda\xi}(K^{\mu}{}_{\mu\nu}-K^{\mu}{}_{\nu\mu})+Ric^{\mu}{}_{\mu}(K_{\xi\nu\lambda}-K_{\nu\xi\lambda})\\
&\quad+Ric^{\mu}{}_{\xi}(K_{\nu\mu\lambda}-K_{\mu\nu\lambda})+Ric_{\lambda\nu}(K^{\mu}{}_{\xi\mu}-K^{\mu}{}_{\mu\xi})+Ric^{\mu}{}_{\nu}(K_{\mu\xi\lambda}-K_{\xi\mu\lambda})]\\
=&\frac{1}{2}[g_{\lambda\nu}Ric^{\mu}{}_{\eta}(K^{\eta}{}_{\xi\mu}-K^{\eta}{}_{\mu\xi})+g_{\lambda\xi}Ric^{\mu}{}_{\eta}(K^{\eta}{}_{\mu\nu}-K^{\eta}{}_{\nu\mu})\\
&\quad+Ric^{\mu}{}_{\xi}(K_{\nu\mu\lambda}-K_{\mu\nu\lambda})+Ric^{\mu}{}_{\nu}(K_{\mu\xi\lambda}-K_{\xi\mu\lambda})\\
&\quad+Ric_{\lambda\nu}(K^{\mu}{}_{\xi\mu}-K^{\mu}{}_{\mu\xi})+Ric_{\lambda\xi}(K^{\mu}{}_{\mu\nu}-K^{\mu}{}_{\nu\mu})].
\end{aligned}$$

The terms from the Bianchi identity involving Ric_* tensor terms, after contracting the indices κ and μ and using the definition of the covariant derivative become

$$\begin{aligned}
&\frac{1}{2}\left\{(K^{\eta}{}_{\mu\xi}-K^{\eta}{}_{\xi\mu})(\epsilon^{\vartheta}{}_{\lambda\eta\nu}Ric_*{}^{\mu}{}_{\vartheta}-\epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_{*\lambda\vartheta})\right.\\
&\quad+(K^{\eta}{}_{\mu\nu}-K^{\eta}{}_{\nu\mu})(\epsilon^{\vartheta}{}_{\lambda\xi\eta}Ric_*{}^{\mu}{}_{\vartheta}-\epsilon^{\vartheta\mu}{}_{\xi\eta}Ric_{*\lambda\vartheta})\\
&\quad\left.-\nabla_{\mu}(\epsilon^{\vartheta}{}_{\lambda\nu\xi}Ric_*{}^{\mu}{}_{\vartheta})+\nabla_{\mu}(\epsilon^{\vartheta\mu}{}_{\nu\xi}Ric_{*\lambda\vartheta})\right\}.
\end{aligned}$$

Putting all these calculations together and using the fact that $Ric^{\mu}{}_{\eta}K^{\eta}{}_{\xi\mu}=0$, the Bianchi identity for curvature becomes

$$\begin{aligned}
0=&\nabla_{\xi}Ric_{\lambda\nu}-\nabla_{\nu}Ric_{\lambda\xi}+g_{\lambda\xi}\nabla_{\mu}Ric^{\mu}{}_{\nu}-g_{\lambda\nu}\nabla_{\mu}Ric^{\mu}{}_{\xi}\\
&+Ric^{\mu}{}_{\eta}(g_{\lambda\xi}K^{\eta}{}_{\mu\nu}-g_{\lambda\nu}K^{\eta}{}_{\mu\xi})+Ric^{\mu}{}_{\xi}(K_{\nu\mu\lambda}-K_{\mu\nu\lambda})+Ric^{\mu}{}_{\nu}(K_{\mu\xi\lambda}-K_{\xi\mu\lambda})\\
&+Ric_{\lambda\nu}(K^{\mu}{}_{\xi\mu}-K^{\mu}{}_{\mu\xi})+Ric_{\lambda\xi}(K^{\mu}{}_{\mu\nu}-K^{\mu}{}_{\nu\mu})\\
&+(K^{\eta}{}_{\mu\xi}-K^{\eta}{}_{\xi\mu})(\epsilon^{\vartheta}{}_{\lambda\eta\nu}Ric_*{}^{\mu}{}_{\vartheta}-\epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_{*\lambda\vartheta})\\
&+(K^{\eta}{}_{\mu\nu}-K^{\eta}{}_{\nu\mu})(\epsilon^{\vartheta}{}_{\lambda\xi\eta}Ric_*{}^{\mu}{}_{\vartheta}-\epsilon^{\vartheta\mu}{}_{\xi\eta}Ric_{*\lambda\vartheta})\\
&-\nabla_{\mu}(\epsilon^{\vartheta}{}_{\lambda\nu\xi}Ric_*{}^{\mu}{}_{\vartheta})+\nabla_{\mu}(\epsilon^{\vartheta\mu}{}_{\nu\xi}Ric_{*\lambda\vartheta})\\
&+2\nabla_{\mu}\mathcal{W}^{\mu}{}_{\lambda\nu\xi}+2\mathcal{W}^{\mu}{}_{\lambda\xi\eta}(K^{\eta}{}_{\nu\mu}-K^{\eta}{}_{\mu\nu})+2\mathcal{W}^{\mu}{}_{\lambda\nu\eta}(K^{\eta}{}_{\mu\xi}-K^{\eta}{}_{\xi\mu}). \qquad \text{(B.6)}
\end{aligned}$$

Now we want to eliminate the $\nabla_{\mu}Ric^{\mu}{}_{\nu}$ term. Contracting the indices λ and ξ in (B.6), we get that

$$\begin{aligned}
0=&\nabla_{\xi}Ric^{\xi}{}_{\nu}-\nabla_{\nu}R+4\nabla_{\mu}Ric^{\mu}{}_{\nu}-\nabla_{\mu}Ric^{\mu}{}_{\nu}+Ric^{\mu}{}_{\eta}(4K^{\eta}{}_{\mu\nu}-K^{\eta}{}_{\mu\nu})\\
&-Ric^{\mu}{}_{\xi}(K^{\xi}{}_{\mu\nu}-K^{\xi}{}_{\nu\mu})+Ric^{\mu}{}_{\nu}(K^{\xi}{}_{\mu\xi}-K^{\xi}{}_{\xi\mu})+Ric^{\xi}{}_{\nu}(K^{\mu}{}_{\xi\mu}-K^{\mu}{}_{\mu\xi})\\
&+Ric^{\xi}{}_{\xi}(K^{\mu}{}_{\mu\nu}-K^{\mu}{}_{\nu\mu})+(K^{\eta}{}_{\mu\xi}-K^{\eta}{}_{\xi\mu})(\epsilon^{\vartheta\xi}{}_{\eta\nu}Ric_*{}^{\mu}{}_{\vartheta}-\epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_*{}^{\xi}{}_{\vartheta})\\
&+2\mathcal{W}^{\mu\xi}{}_{\nu\eta}(K^{\eta}{}_{\mu\xi}-K^{\eta}{}_{\xi\mu}).
\end{aligned}$$

Hence we can express the ∇Ric term as

$$\nabla_\mu Ric^\mu{}_\nu = -\frac{1}{2}Ric^\mu{}_\eta K^\eta{}_{\mu\nu} - \frac{1}{2}Ric^\beta{}_\nu(K^\mu{}_{\beta\mu} - K^\mu{}_{\mu\beta}) - \frac{1}{2}\mathcal{W}^{\mu\alpha}{}_{\nu\eta}(K^\eta{}_{\mu\alpha} - K^\eta{}_{\alpha\mu})$$
$$- \frac{1}{2}K^\eta{}_{\mu\beta}(\epsilon^{\vartheta\beta}{}_{\eta\nu}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_*{}^\beta{}_\vartheta). \tag{B.7}$$

Substituting (B.7) into (B.6), we get that

$$\begin{aligned}
\nabla_\mu \mathcal{W}^\mu{}_{\lambda\nu\xi} = &-\frac{1}{2}[\nabla_\xi Ric_{\lambda\nu} - \nabla_\nu Ric_{\lambda\xi} + \nabla_\mu(\epsilon^{\vartheta\mu}{}_{\nu\xi}Ric_{*\lambda\vartheta} - \epsilon^\vartheta{}_{\lambda\nu\xi}Ric_*{}^\mu{}_\vartheta)\\
&+ Ric^\mu{}_\xi(K_{\nu\mu\lambda} - K_{\mu\nu\lambda}) + Ric^\mu{}_\nu(K_{\mu\xi\lambda} - K_{\xi\mu\lambda}) + Ric_{\lambda\nu}(K^\mu{}_{\xi\mu} - K^\mu{}_{\mu\xi})\\
&+ Ric_{\lambda\xi}(K^\mu{}_{\mu\nu} - K^\mu{}_{\nu\mu}) + (K^\eta{}_{\mu\xi} - K^\eta{}_{\xi\mu})(\epsilon^\vartheta{}_{\lambda\eta\nu}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_{*\lambda\vartheta})\\
&+ (K^\eta{}_{\mu\nu} - K^\eta{}_{\nu\mu})(\epsilon^\vartheta{}_{\lambda\xi\eta}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\xi\eta}Ric_{*\lambda\vartheta})]\\
&+ \frac{1}{4}Ric^\mu{}_\eta(g_{\lambda\xi}K^\eta{}_{\mu\nu} - g_{\lambda\nu}K^\eta{}_{\mu\xi}) + \frac{1}{4}(g_{\lambda\xi}Ric^\beta{}_\nu - g_{\lambda\nu}Ric^\beta{}_\xi)(K^\mu{}_{\beta\mu} - K^\mu{}_{\mu\beta})\\
&+ \frac{1}{4}g_{\lambda\xi}K^\eta{}_{\mu\beta}(\epsilon^{\vartheta\beta}{}_{\eta\nu}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\eta\nu}Ric_*{}^\beta{}_\vartheta)\\
&- \frac{1}{4}g_{\lambda\nu}K^\eta{}_{\mu\beta}(\epsilon^{\vartheta\beta}{}_{\eta\xi}Ric_*{}^\mu{}_\vartheta - \epsilon^{\vartheta\mu}{}_{\eta\xi}Ric_*{}^\beta{}_\vartheta)\\
&+ \frac{1}{4}(g_{\lambda\xi}\mathcal{W}^{\mu\alpha}{}_{\nu\eta} - g_{\lambda\nu}\mathcal{W}^{\mu\alpha}{}_{\xi\eta})(K^\eta{}_{\mu\alpha} - K^\eta{}_{\alpha\mu})\\
&- \mathcal{W}^\mu{}_{\lambda\xi\eta}(K^\eta{}_{\nu\mu} - K^\eta{}_{\mu\nu}) - \mathcal{W}^\mu{}_{\lambda\nu\eta}(K^\eta{}_{\mu\xi} - K^\eta{}_{\xi\mu}).
\end{aligned} \tag{B.8}$$

Remark B.2.1. Note that the equation (B.8) represents the Bianchi identity for curvature without using any assumptions on the nature of the torsion.

Now we apply assumption (ii) that the torsion is purely axial and the relationship between the Levi-Civita and Weyl tensors given in Lemma 1.4.11. Equation (B.7) now becomes

$$\nabla_\mu Ric^\mu{}_\nu = \epsilon^{\vartheta\beta}{}_{\nu\eta}K^\eta{}_{\mu\beta}Ric_*{}^\mu{}_\vartheta, \tag{B.9}$$

and equation (B.8) becomes

$$\begin{aligned}
\nabla_\mu \mathcal{W}^\mu{}_{\lambda\nu\xi} = &-\frac{1}{2}\left[\nabla_\xi Ric_{\lambda\nu} - \nabla_\nu Ric_{\lambda\xi} + 2Ric^\mu{}_\xi K_{\nu\mu\lambda} + 2Ric^\mu{}_\nu K_{\mu\xi\lambda}\right.\\
&\left.+\epsilon^{\vartheta\mu}{}_{\nu\xi}\nabla_\mu Ric_{*\lambda\vartheta} - \epsilon^\vartheta{}_{\lambda\nu\xi}\nabla_\mu Ric_*{}^\mu{}_\vartheta\right]\\
&- \epsilon^\vartheta{}_{\lambda\eta\nu}K^\eta{}_{\mu\xi}Ric_*{}^\mu{}_\vartheta + \epsilon^{\vartheta\mu}{}_{\eta\nu}K^\eta{}_{\mu\xi}Ric_{*\lambda\vartheta}\\
&- \epsilon^\vartheta{}_{\lambda\xi\eta}K^\eta{}_{\mu\nu}Ric_*{}^\mu{}_\vartheta + \epsilon^{\vartheta\mu}{}_{\xi\eta}K^\eta{}_{\mu\nu}Ric_{*\lambda\vartheta}\\
&+ \frac{1}{2}g_{\lambda\xi}\epsilon^{\vartheta\zeta}{}_{\eta\nu}K^\eta{}_{\mu\zeta}Ric_*{}^\mu{}_\vartheta - \frac{1}{2}g_{\lambda\nu}\epsilon^{\vartheta\zeta}{}_{\eta\xi}K^\eta{}_{\mu\zeta}Ric_*{}^\mu{}_\vartheta\\
&- 2\mathcal{W}^\mu{}_{\lambda\xi\eta}K^\eta{}_{\nu\mu} - 2\mathcal{W}^\mu{}_{\lambda\nu\eta}K^\eta{}_{\mu\xi}.
\end{aligned} \tag{B.10}$$

Appendix C

Explicit Variations of Some Quadratic Forms on Curvature

In this appendix we provide the explicit variations of certain quadratic forms on curvature which are used in this thesis. We provide only the variations with respect to the metric following Proposition A.1.1.

C.1 Variation of $\int Ric_{\mu\nu}Ric^{\mu\nu}$

Since

$$\frac{\delta}{\delta g}\int Ric_{\mu\nu}Ric^{\mu\nu}\sqrt{|\det g|} = \int \delta\left(Ric_{\mu\nu}Ric_{\mu'\nu'}g^{\mu\mu'}g^{\nu\nu'}\sqrt{|\det g|}\right),$$

we get that

$$\begin{aligned}
&\frac{\delta}{\delta g}\int Ric_{\mu\nu}Ric^{\mu\nu}\sqrt{|\det g|} \\
&= \int (\delta g_{\alpha\beta})\, Ric_{\mu\nu}Ric_{\mu'\nu'}\left(-g^{\nu\nu'}g^{\mu\alpha}g^{\beta\mu'} - g^{\mu\mu'}g^{\nu\alpha}g^{\beta\nu'} + \frac{1}{2}g^{\mu\mu'}g^{\nu\nu'}g^{\alpha\beta}\right) \\
&= \int (\delta g_{\alpha\beta})\left(-Ric^{\alpha}{}_{\nu}Ric^{\beta\nu} - Ric_{\mu}{}^{\alpha}Ric^{\mu\beta} + \frac{1}{2}Ric_{\mu\nu}Ric^{\mu\nu}g^{\alpha\beta}\right).
\end{aligned}$$

Remark C.1.1. Under the assumption that the spacetime is metric compatible, Ricci curvature becomes symmetric and the last equation simplifies to

$$\frac{\delta}{\delta g}\int Ric_{\mu\nu}Ric^{\mu\nu}\sqrt{|\det g|} = -2\int (\delta g_{\alpha\beta})\left(Ric_{\mu}{}^{\alpha}Ric^{\mu\beta} - \frac{1}{4}Ric_{\mu\nu}Ric^{\mu\nu}g^{\alpha\beta}\right).$$

C.2 Variation of $\int Ric^{(2)}_{\mu\nu} Ric_{\mu\nu}$

Using Proposition A.1.1, we get that

$$\begin{aligned}
&\frac{\delta}{\delta g}\int Ric^{(2)\kappa}{}_{\nu} Ric_{\kappa}{}^{\nu}\sqrt{|\det g|} \\
&\quad = \frac{\delta}{\delta g}\int R^{\kappa\lambda}{}_{\lambda\nu} R^{\mu}{}_{\kappa\mu}{}^{\nu}\sqrt{|\det g|} = \frac{\delta}{\delta g}\int R^{\kappa}{}_{\lambda'\lambda\nu} R^{\mu}{}_{\kappa\mu\nu'} g^{\lambda\lambda'} g^{\nu\nu'}\sqrt{|\det g|} \\
&\quad = \int (\delta g_{\alpha\beta}) R^{\kappa}{}_{\lambda'\lambda\nu} R^{\mu}{}_{\kappa\mu\nu'}\left(-g^{\lambda\alpha} g^{\beta\lambda'} g^{\nu\nu'}\sqrt{|\det g|} - g^{\lambda\lambda'} g^{\nu\alpha} g^{\beta\nu'}\sqrt{|\det g|}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{1}{2} g^{\lambda\lambda'} g^{\nu\nu'} g^{\alpha\beta}\sqrt{|\det g|}\right) \\
&\quad = \int (\delta g_{\alpha\beta})\left(-R^{\kappa\alpha\beta}{}_{\nu} Ric_{\kappa}{}^{\nu} - Ric^{(2)\kappa\alpha} Ric_{\kappa}{}^{\beta} + \frac{1}{2} g^{\alpha\beta} Ric^{(2)\kappa}{}_{\nu} Ric_{\kappa}{}^{\nu}\right).
\end{aligned}$$

Remark C.2.1. Under the assumption that the spacetime is metric compatible, $Ric^{(2)} = -Ric$ and the last equation simplifies to

$$\begin{aligned}
&\frac{\delta}{\delta g}\int Ric^{(2)\kappa}{}_{\nu} Ric_{\kappa}{}^{\nu}\sqrt{|\det g|} \\
&\quad = \int (\delta g_{\alpha\beta})\left(-R^{\kappa\alpha\beta}{}_{\nu} Ric_{\kappa}{}^{\nu} + Ric^{\kappa\alpha} Ric_{\kappa}{}^{\beta} - \frac{1}{2} g^{\alpha\beta} Ric^{\kappa}{}_{\nu} Ric_{\kappa}{}^{\nu}\right).
\end{aligned}$$

C.3 Variation of $\int Ric^{(2)}_{\mu\nu} Ric^{(2)\mu\nu}$

Since $Ric^{(2)}_{\kappa\nu} = R_{\kappa}{}^{\mu}{}_{\mu\nu}$, we have that

$$\begin{aligned}
&\frac{\delta}{\delta g}\int Ric^{(2)}_{\kappa\nu} Ric^{(2)\kappa\nu} = \frac{\delta}{\delta g}\int Ric^{(2)}_{\kappa\nu} Ric^{(2)\kappa\nu}\sqrt{|\det g|} \\
&\quad = \frac{\delta}{\delta g}\int R_{\kappa}{}^{\mu}{}_{\mu\nu} R^{\kappa\lambda}{}_{\lambda}{}^{\nu}\sqrt{|\det g|} = \frac{\delta}{\delta g}\int R^{\kappa'}{}_{\mu'\mu\nu} R^{\kappa}{}_{\lambda'\lambda\nu'} g_{\kappa\kappa'} g^{\mu\mu'} g^{\lambda\lambda'} g^{\nu\nu'}\sqrt{|\det g|}.
\end{aligned}$$

Using Proposition A.1.1, we get that

$$\begin{aligned}
\frac{\delta}{\delta g}\int Ric^{(2)}_{\kappa\nu} Ric^{(2)\kappa\nu} &= \int (\delta g_{\alpha\beta})\left(R^{\beta\mu}{}_{\mu\nu} R^{\alpha\lambda}{}_{\lambda}{}^{\nu} - R^{\kappa\beta\alpha\nu} R_{\kappa}{}^{\lambda}{}_{\lambda\nu}\right. \\
&\left. - R_{\kappa}{}^{\mu}{}_{\mu\nu} R^{\kappa\beta\alpha\nu} - R_{\kappa}{}^{\mu}{}_{\mu}{}^{\alpha} R^{\kappa\lambda}{}_{\lambda}{}^{\beta} + \frac{1}{2} R_{\kappa}{}^{\mu}{}_{\mu\nu} R^{\kappa\lambda}{}_{\lambda}{}^{\nu} g^{\alpha\beta}\right) \\
&= \int (\delta g_{\alpha\beta})\left(Ric^{(2)\beta}{}_{\nu} Ric^{(2)\alpha\nu} - Ric^{(2)}{}_{\kappa}{}^{\alpha} Ric^{(2)\kappa\beta}\right. \\
&\left. - 2 Ric^{(2)}{}_{\kappa\nu} R^{\kappa\beta\alpha\nu} + \frac{1}{2} g^{\alpha\beta} Ric^{(2)}{}_{\kappa\nu} Ric^{(2)\kappa\nu}\right).
\end{aligned}$$

Remark C.3.1. Under the assumption that the spacetime is metric compatible, Ricci curvature is symmetric and $Ric^{(2)} = -Ric$, so the last equation simplifies to

$$\frac{\delta}{\delta g}\int Ric^{(2)}_{\kappa\nu} Ric^{(2)\kappa\nu}\sqrt{|\det g|} = \int (\delta g_{\alpha\beta})\left(2Ric_{\kappa\nu}R^{\kappa\beta\alpha\nu} + \frac{1}{2}g^{\alpha\beta}Ric_{\kappa\nu}Ric^{\kappa\nu}\right).$$

Appendix D

Detailed calculations of the asymptotic coefficients

In this appendix we provide detailed calculations of the formulae for the asymptotic coefficients (3.70), (3.71), (3.72) and (3.73) from Theorem 3.5.7 which correspond to the eigenvalues $\lambda = \pm 1$. We will use the perturbation theory which is described in Section 3.4 and the explicitly derived formulae for the asymptotic coefficients given by (3.52), (3.53). We will also use the concept of the pseudoinverse of the massless Dirac operator whose construction is given in Section 3.4.1.

D.1 The calculations of the $\lambda_\pm^{(1)}$ coefficients

We first want to show formulae (3.70), (3.71) from Theorem 3.5.7. Using the formula for the eigenvectors (3.77), (3.79) corresponding to the eigenvalues $\lambda = 1$ and $\lambda = -1$ respectively, as well as formula (3.75) for the differential operator $W_{1/2}^{(1)}$, integrating by parts, we get that the equation (3.52) for the eigenvalues $\lambda = \pm 1$ becomes

$$
\begin{aligned}
\lambda_+^{(1)} &= \langle W_{1/2}^{(1)} v^{(0)}, v^{(0)} \rangle = \int_0^{2\pi} [v^{(0)}]^* W_{1/2}^{(1)} v^{(0)} dx^1 \\
&= \frac{i}{4} \int_0^{2\pi} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} \frac{d}{dx^1} v^{(0)} dx^1 \\
&- \frac{i}{4} \int_0^{2\pi} \frac{d}{dx^1} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} v^{(0)} dx^1 \\
&= -\frac{1}{4\pi} \int_0^{2\pi} {h_1}^1(x^1) dx^1 = -\frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} {h_1}^1(x^1) dx^1 = -\frac{1}{2} \widehat{h}_{11}(0),
\end{aligned} \tag{D.1}
$$

$$
\begin{aligned}
\lambda_-^{(1)} &= \langle W_{1/2}^{(1)} v^{(0)}, v^{(0)} \rangle = \int_0^{2\pi} [v^{(0)}]^* W_{1/2}^{(1)} v^{(0)} dx^1 \\
&= \frac{i}{4} \int_0^{2\pi} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} \frac{d}{dx^1} v^{(0)} dx^1 \\
&- \frac{i}{4} \int_0^{2\pi} \frac{d}{dx^1} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} v^{(0)} dx^1 \\
&= \frac{1}{4\pi} \int_0^{2\pi} {h_1}^1(x^1) dx^1 = \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} {h_1}^1(x^1) dx^1 = \frac{1}{2} \widehat{h}_{11}(0),
\end{aligned}
\tag{D.2}
$$

where $\widehat{h}(m)$ denotes the Fourier coefficient of the function $h(x^1)$, see Definition 3.5.1.

D.2 The calculations of the $\lambda_\pm^{(2)}$ coefficients

First, we will prove the formula (3.72) from Theorem 3.5.7 for the coefficient $\lambda_+^{(2)}$ in the asymptotic expansion (3.68) of the eigenvalue $\lambda = 1$. We will separate the calculation of this coefficient into several parts, for sake of simplicity and readability, using the explicit formula (3.53). Let us therefore first evaluate the term $\langle W_{1/2}^{(2)} v^{(0)}, v^{(0)} \rangle$. Using the formula for the eigenvector (3.77) corresponding to the eigenvalue $\lambda = 1$, as well as formula (3.76) for the differential operator $W_{1/2}^{(2)}$, integrating by parts we get that

$$
\begin{aligned}
\langle W_{1/2}^{(2)} v^{(0)}, v^{(0)} \rangle &= \int_0^{2\pi} [v^{(0)}]^* W_{1/2}^{(2)} v^{(0)} dx^1 \\
&= -\frac{3i}{16} \int_0^{2\pi} [v^{(0)}]^* \begin{pmatrix} (h^2)_3{}^1 & (h^2)_1{}^1 - i(h^2)_2{}^1 \\ (h^2)_1{}^1 + i(h^2)_2{}^1 & -(h^2)_3{}^1 \end{pmatrix} \frac{d}{dx^1} v^{(0)} dx^1 \\
&- \frac{3i}{16} \int_0^{2\pi} [v^{(0)}]^* \frac{d}{dx^1} \begin{pmatrix} (h^2)_3{}^1 & (h^2)_1{}^1 - i(h^2)_2{}^1 \\ (h^2)_1{}^1 + i(h^2)_2{}^1 & -(h^2)_3{}^1 \end{pmatrix} v^{(0)} dx^1 \\
&+ \frac{i}{16} \int_0^{2\pi} [v^{(0)}]^* \begin{pmatrix} {k_3}^1 & {k_1}^1 - i{k_2}^1 \\ {k_1}^1 + i{k_2}^1 & -{k_3}^1 \end{pmatrix} \frac{d}{dx^1} v^{(0)} dx^1 \\
&+ \frac{i}{16} \int_0^{2\pi} [v^{(0)}]^* \frac{d}{dx^1} \begin{pmatrix} {k_3}^1 & {k_1}^1 - i{k_2}^1 \\ {k_1}^1 + i{k_2}^1 & -{k_3}^1 \end{pmatrix} v^{(0)} dx^1 \\
&+ \int_0^{2\pi} [v^{(0)}]^* \left(-\frac{1}{16} \right) \varepsilon_{\beta\gamma 1} h_{\alpha\beta} \frac{dh_{\alpha\gamma}}{dx^1} I v^{(0)} \\
&= \frac{3}{8} \widehat{(h^2)}_{11}(0) - \frac{1}{8} \widehat{k}_{11}(0) - \frac{i}{16} \varepsilon_{\beta\gamma 1} \sum_{m_1 \in \mathbb{Z} \setminus \{0\}} m_1 \overline{\widehat{h}_{\alpha\beta}(m_1)} \widehat{h}_{\alpha\gamma}(m_1).
\end{aligned}
\tag{D.3}
$$

Remark D.2.1. We simplified equation (D.3) using the Fourier coefficients, see Definition 3.5.1, and the Parseval's formula

$$\frac{1}{2\pi}\int_0^{2\pi} p(x)\overline{q(x)}dx = \sum_{m\in\mathbb{Z}} \widehat{p}(m)\overline{\widehat{q}(m)}.$$

Secondly, we will evaluate the term $\langle (W^{(1)}_{1/2} - \lambda^{(1)})Q(W^{(1)}_{1/2} - \lambda^{(1)})v^{(0)}, v^{(0)}\rangle$. The pseudoinverse operator, see Section 3.4.1, corresponding to the operator $W_{1/2} - I$ is given by

$$\begin{aligned} Q_+ = \frac{1}{4\pi}\sum_{m\in\mathbb{Z}\backslash\{1\}} \frac{1}{m-1}\Bigg[& e^{imx^1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\int_0^{2\pi} e^{-imy^1}(\,\cdot\,)dy^1 \\ & +e^{-imx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\int_0^{2\pi} e^{imy^1}(\,\cdot\,)dy^1\Bigg]. \end{aligned} \tag{D.4}$$

Using (3.75), (3.77) and (D.1), we have that

$$\begin{aligned} (W^{(1)}_{1/2} - \lambda^{(1)}_+)v^{(0)} = & -\frac{1}{4\sqrt{\pi}}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} e^{ix^1} \\ & + \frac{i}{8\sqrt{\pi}}e^{ix^1}\frac{d}{dx^1}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} + \frac{\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{ix^1}. \end{aligned} \tag{D.5}$$

Using the explicit formula for the pseudoinverse (D.4), now we will evaluate the term $Q_+((W^{(1)}_{1/2} - \lambda^{(1)}_+)v^{(0)})$ in three parts. First we will act with the pseudoinverse Q_+ on the first term on the RHS of equation (D.5). Using the well known property of the Fourier coefficient that $\widehat{h}(-m) = \overline{\widehat{h}(m)}$, we obtain

$$\begin{aligned} & Q_+\left(-\frac{1}{4\sqrt{\pi}}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} e^{ix^1}\right) \\ & = -\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\backslash\{1\}}\frac{1}{m-1} \\ & \Bigg[e^{imx^1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} e^{-i(m-1)y^1}dy^1 \\ & + e^{-imx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} e^{-i(-(m+1))y^1}dy^1\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{8\sqrt{\pi}} \sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1} \\
&\quad \Bigg[e^{imx^1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{h}_{11}(m-1) - i\widehat{h}_{21}(m-1) + \widehat{h}_{31}(m-1) \\ \widehat{h}_{11}(m-1) + i\widehat{h}_{21}(m-1) - \widehat{h}_{31}(m-1) \end{pmatrix} \\
&\quad + e^{-imx^1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \overline{\widehat{h}_{11}(m+1)} - i\overline{\widehat{h}_{21}(m+1)} + \overline{\widehat{h}_{31}(m+1)} \\ \overline{\widehat{h}_{11}(m+1)} + i\overline{\widehat{h}_{21}(m+1)} - \overline{\widehat{h}_{31}(m+1)} \end{pmatrix} \Bigg] \\
&= -\frac{1}{4\sqrt{\pi}} \sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1} \\
&\quad \begin{pmatrix} \widehat{h}_{11}(m-1)e^{imx^1} - i\,\overline{\widehat{h}_{21}(m+1)}e^{-imx^1} + \overline{\widehat{h}_{31}(m+1)}e^{-imx^1} \\ \widehat{h}_{11}(m-1)e^{imx^1} + i\,\overline{\widehat{h}_{21}(m+1)}e^{-imx^1} - \overline{\widehat{h}_{31}(m+1)}e^{-imx^1} \end{pmatrix}.
\end{aligned}
$$

Secondly, we act with the pseudoinverse Q_+ on the second term on the RHS of (D.5). Integrating by parts, we obtain

$$
\begin{aligned}
&Q_+ \left(\frac{i}{8\sqrt{\pi}} e^{ix^1} \frac{d}{dx^1} \begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} \right) = -\frac{1}{16\sqrt{\pi}} \sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1} \\
&\quad \Bigg[e^{imx^1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{m-1}{2\pi} \int_0^{2\pi} e^{-i(m-1)y^1} \begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} dy^1 \\
&\quad - e^{-imx^1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{m+1}{2\pi} \int_0^{2\pi} e^{-i(-(m+1))y^1} \begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} dy^1 \Bigg] \\
&= -\frac{1}{16\sqrt{\pi}} \sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1} \\
&\quad \Bigg[(m-1)e^{imx^1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{h}_{11}(m-1) - i\widehat{h}_{21}(m-1) + \widehat{h}_{31}(m-1) \\ \widehat{h}_{11}(m-1) + i\widehat{h}_{21}(m-1) - \widehat{h}_{31}(m-1) \end{pmatrix} \\
&\quad - (m+1)\, e^{-imx^1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \overline{\widehat{h}_{11}(m+1)} - i\overline{\widehat{h}_{21}(m+1)} + \overline{\widehat{h}_{31}(m+1)} \\ \overline{\widehat{h}_{11}(m+1)} + i\overline{\widehat{h}_{21}(m+1)} - \overline{\widehat{h}_{31}(m+1)} \end{pmatrix} \Bigg] \\
&= -\frac{1}{8\sqrt{\pi}} \sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1} \\
&\quad \begin{pmatrix} (m-1)\widehat{h}_{11}(m-1)e^{imx^1} + i(m+1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1} - (m+1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1} \\ (m-1)\widehat{h}_{11}(m-1)e^{imx^1} - i(m+1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1} + (m+1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1} \end{pmatrix}.
\end{aligned}
$$

Thirdly, we act with the pseudoinverse Q_+ on the third and final term on the RHS of (D.5) to obtain

$$Q_+\left(\frac{\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix}1\\1\end{pmatrix}e^{ix^1}\right)$$
$$=\frac{\widehat{h}_{11}(0)}{16\pi\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\left[e^{imx^1}\begin{pmatrix}1&1\\1&1\end{pmatrix}\int_0^{2\pi}\begin{pmatrix}1\\1\end{pmatrix}e^{i(1-m)y^1}dy^1\right.$$
$$\left.+e^{-imx^1}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\int_0^{2\pi}\begin{pmatrix}1\\1\end{pmatrix}e^{i(m+1)y^1}dy^1\right]. \quad \text{(D.6)}$$

For $m\in\mathbb{Z}$ we have that

$$\int_0^{2\pi}e^{i(1-m)y^1}dy^1=\begin{cases}2\pi, & m=1,\\ 0, & m\neq 1,\end{cases},\quad \int_0^{2\pi}e^{i(m+1)y^1}dy^1=\begin{cases}2\pi, & m=-1,\\ 0, & m\neq -1,\end{cases}$$

we get that the sum (D.6) only makes sense if $m=-1$. Hence

$$Q_+\left(\frac{\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix}1\\1\end{pmatrix}e^{ix^1}\right)=-\frac{\widehat{h}_{11}(0)}{32\pi\sqrt{\pi}}e^{ix^1}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}2\pi\\2\pi\end{pmatrix}=0.$$

Putting the three above calculations together, we get that

$$Q_+((A^{(1)}-\lambda_+^{(1)})v^{(0)})=-\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}$$
$$\begin{pmatrix}(m+1)\widehat{h}_{11}(m-1)e^{imx^1}+i(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}-(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\\(m+1)\widehat{h}_{11}(m-1)e^{imx^1}-i(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}+(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\end{pmatrix}. \quad \text{(D.7)}$$

Now we will calculate the part $((W_{1/2}^{(1)}-\lambda_+^{(1)})Q_+((W_{1/2}^{(1)}-\lambda_+^{(1)})v^{(0)})$. Since

$$\frac{d(Q_+((A^{(1)}-\lambda_+^{(1)})v^{(0)}))}{dx^1}=-\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}$$
$$\begin{pmatrix}im(m+1)\widehat{h}_{11}(m-1)e^{imx^1}+m(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}+im(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\\im(m+1)\widehat{h}_{11}(m-1)e^{imx^1}-m(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}-im(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\end{pmatrix}$$

and

$$\frac{d}{dx^1}\left(\begin{pmatrix}h_{31}&h_{11}-ih_{21}\\h_{11}+ih_{21}&-h_{31}\end{pmatrix}Q_+((A^{(1)}-\lambda_+^{(1)})v^{(0)})\right)$$
$$=-\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\left[(m+1)\widehat{h}_{11}(m-1)\frac{d}{dx^1}\begin{pmatrix}h_{11}-ih_{21}+h_{31}\\h_{11}+ih_{21}-h_{31}\end{pmatrix}e^{imx^1}\right.$$
$$+i(m-1)\overline{\widehat{h}_{21}(m+1)}\frac{d}{dx^1}\begin{pmatrix}h_{11}-ih_{21}-h_{31}\\h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}$$
$$\left.-(m-1)\overline{\widehat{h}_{31}(m+1)}\frac{d}{dx^1}\begin{pmatrix}h_{11}-ih_{21}-h_{31}\\h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}\right],$$

using equations (3.75), (D.1) and (D.7), we get that

$$
\begin{aligned}
&(W^{(1)}_{1/2}-\lambda^{(1)}_{+})Q_{+}((W^{(1)}_{1/2}-\lambda^{(1)}_{+})v^{(0)})\\
&\quad=-\frac{i}{32\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\\
&\qquad\Bigg[im(m+1)\widehat{h}_{11}(m-1)e^{imx^1}\begin{pmatrix} h_{11}-ih_{21}+h_{31}\\ h_{11}+ih_{21}-h_{31}\end{pmatrix}\\
&\qquad+m(m-1)\overline{\widehat{h}_{21}(m+1)}\begin{pmatrix} -h_{11}+ih_{21}+h_{31}\\ h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}\\
&\qquad+im(m-1)\overline{\widehat{h}_{31}(m+1)}\begin{pmatrix} -h_{11}+ih_{21}+h_{31}\\ h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}\Bigg]\\
&\quad-\frac{i}{32\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\\
&\qquad\Bigg[(m+1)\widehat{h}_{11}(m-1)\frac{d}{dx^1}\begin{pmatrix} h_{11}-ih_{21}+h_{31}\\ h_{11}+ih_{21}-h_{31}\end{pmatrix}e^{imx^1}\\
&\qquad+i(m-1)\overline{\widehat{h}_{21}(m+1)}\frac{d}{dx^1}\begin{pmatrix} -h_{11}+ih_{21}+h_{31}\\ h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}\\
&\qquad-(m-1)\overline{\widehat{h}_{31}(m+1)}\frac{d}{dx^1}\begin{pmatrix} -h_{11}+ih_{21}+h_{31}\\ h_{11}+ih_{21}+h_{31}\end{pmatrix}e^{-imx^1}\Bigg]\\
&\quad-\frac{\widehat{h}_{11}(0)}{16\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}\\
&\qquad\begin{pmatrix} (m+1)\widehat{h}_{11}(m-1)e^{imx^1}+i(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}-(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\\ (m+1)\widehat{h}_{11}(m-1)e^{imx^1}-i(m-1)\overline{\widehat{h}_{21}(m+1)}e^{-imx^1}+(m-1)\overline{\widehat{h}_{31}(m+1)}e^{-imx^1}\end{pmatrix}.
\end{aligned}
$$

Finally, the second term $\langle (W^{(1)}_{1/2}-\lambda^{(1)}_{+})Q_{+}(W^{(1)}_{1/2}-\lambda^{(1)}_{+})v^{(0)}, v^{(0)}\rangle$, using (3.13), becomes

$$
\begin{aligned}
&\langle (W^{(1)}_{1/2}-\lambda^{(1)}_{+})Q_{+}((W^{(1)}_{1/2}-\lambda^{(1)}_{+})v^{(0)}), v^{(0)}\rangle\\
&\quad=\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}}\frac{1}{m-1}(m+1)^2\,\widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)}\\
&\quad+\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}}(m-1)\,\Big(\widehat{h}_{31}(m+1)+i\widehat{h}_{21}(m+1)\Big)\Big(\overline{\widehat{h}_{31}(m+1)}-i\overline{\widehat{h}_{21}(m+1)}\Big). \qquad (D.8)
\end{aligned}
$$

Combining equations (D.3) and (D.8) , we get the formula (3.53), for the coefficient $\lambda^{(2)}_{+}$ is given by

$$
\begin{aligned}
\lambda_{+}^{(2)} = {}& \frac{3}{8}\widehat{(h^2)}_{11}(0) - \frac{1}{8}\widehat{k}_{11}(0) - \frac{i}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} \frac{1}{m-1}(m+1)^2\,\widehat{h}_{11}(m-1)\overline{\widehat{h}_{11}(m-1)} \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{1\}} (m-1)\,\left(\widehat{h}_{31}(m+1) + i\widehat{h}_{21}(m+1)\right)\left(\overline{\widehat{h}_{31}(m+1)} - i\overline{\widehat{h}_{21}(m+1)}\right).
\end{aligned}
$$

Remark D.2.2. Using the eigenvector (3.79) corresponding to the eigenvalue $\lambda = -1$ of the massless Dirac operator and the pseudoinverse operator (3.80) of the operator $W_{1/2} + I$, analogously to the above calculations performed for the eigenvector (3.77) corresponding to the eigenvalue $\lambda = 1$, we get that the formula (3.53), for the coefficient $\lambda_{-}^{(2)}$ is given by

$$
\begin{aligned}
\lambda_{-}^{(2)} = {}& -\frac{3}{8}\widehat{(h^2)}_{11}(0) + \frac{1}{8}\widehat{k}_{11}(0) - \frac{i}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} \frac{1}{m+1}(m-1)^2\,\widehat{h}_{11}(m+1)\overline{\widehat{h}_{11}(m+1)} \\
& -\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-1\}} (m+1)\,\left(\widehat{h}_{31}(m-1) + i\widehat{h}_{21}(m-1)\right)\left(\overline{\widehat{h}_{31}(m-1)} - i\overline{\widehat{h}_{21}(m-1)}\right).
\end{aligned}
$$

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